Mathematical Reasoning — Final Exam
MAT 312, Fall 2017 — D. Ivanšić

Name: ____________________________

Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) $4 > 2$ and $4 \geq -2$.

2. (3pts) For every natural number n, if n is divisible by a natural number greater than n, then n is even.

3. (3pts) (universal set=**Z**) $3x^2 + 14x = 5$.

Negate the following statements. Write English sentences, but you may assist yourself with symbols, if necessary.

4. (3pts) If a real number is less than 1, then its reciprocal is greater than 1.

5. (4pts) For every $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

6. (10pts) Use previously proven logical equivalences to prove the equivalence $P \implies (Q \wedge R) \equiv (P \implies Q) \wedge (P \implies R)$. Do not use a truth table.

7. (16pts) Consider the statement: if x is irrational, then $\sqrt{x}$ is irrational.

a) State the converse and prove or disprove it:

b) State the contrapositive and prove or disprove it:

c) Is this statement true or false (justify): x is irrational if and only if $\sqrt{x}$ is irrational?

8. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\exists y \in \mathbf{R})(x + y^2 = 9)$$

a) If $x = -3$, is the statement true?
b) If $x = 12$, is the statement true?
c) Find the truth set (the x's) of the above statement.

9. (14pts) Prove using induction: for every integer $n \geq 2$, $4^2 + 4^3 + \cdots + 4^n = \frac{4^{n+1} - 16}{3}$.

10. (14pts) We have shown on homework: for every integer n, if n^2 is even, then n is even. Use this proposition to show directly that $\sqrt{8}$ is irrational, that is, **without** using the fact that $\sqrt{2}$ is irrational. (Do **not** use the statement "If n^2 is divisible by 8, then n is divisible by 8," because it is not true.)

11. (12pts) Prove that for all real numbers x and y, $x^2 + y^2 + 2 \geq 2y - 2x$.

12. (12pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the three sets, two are equal. Use set algebra to show they are equal.

$(A \cup B) - C$ $\quad\quad$ $(A - B) \cup (B - C)$ $\quad\quad$ $(A - C) \cup (B - C)$

13. (12pts) Let $A = \{k \in \mathbf{Z} \mid k \equiv 3 \pmod 5\}$ and $B = \{k \in \mathbf{Z} \mid k^2 - k \equiv 1 \pmod 5\}$.
a) Prove $A \subseteq B$.
b) Prove $B \subseteq A$ by proving the equivalent: $A^c \subseteq B^c$.

14. (16pts) Let $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be given by $f(x, y) = y + x^2$.
a) Is f surjective? Justify.
b) Is f injective? Justify.
c) Determine the set of preimages of 5. List at least three elements of this set and illustrate it in the plane.

15. (5pts) Draw an arrow diagram between the provided two copies of **Z** that illustrates a function $f : \mathbf{Z} \to \mathbf{Z}$ that is an injection and is not a surjection (pattern needs to be obvious).

$$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$$

$$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$$

16. (12pts) Let A, B be subsets of a universal set U. Prove that $A \subseteq B$ if and only if $A \cup B = B$.

Bonus. (10pts) Consider the general quadratic equation $ax^2 + bx + c = 0$. Prove the following statement: if $a > 0$, $b < 0$ and $c > 0$ and the equation has a real solution, then both solutions are positive.