Mathematical Reasoning — Final Exam
MAT 312, Fall 2020 — D. Ivanšić

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Show all your work!

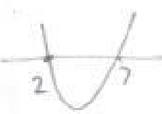
Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) If $x \in \mathbf{R}$ and $x^2 < -3$, then $x^2 < -1$.

True. $F \Rightarrow F$ is a true statement

2. (4pts) (universal set=$\mathbf{R}$) $x^2 - 9x + 14 < 0$.

open sentence $(x-7)(x-2)<0$



Truth set: $(2,7)$

3. (2pts) For every $x \in \mathbf{R}$, $3x + 7 \geq 0$

False. Take $x=-3$, $3(-3)+7=-2 \not\geq 0$

statement.

Negate the following statements. Write English sentences, but you may assist yourself with symbols, if necessary.

4. (3pts) If the product of two real numbers is positive, then both numbers are positive.

The product of two real numbers is positive and at least one of them is not positive

5. (4pts) There exists an $x \in A$, such that x is even and $x \equiv 1 \pmod 3$.

For every $x \in A$, x is odd or $x \not\equiv 1 \pmod 3$

6. (10pts) Use previously proven logical equivalences to prove the equivalence $(P \Longrightarrow Q) \vee (R \Longrightarrow Q) \equiv (P \wedge R) \Longrightarrow Q$. Do not use a truth table.

$$(P \Rightarrow Q) \vee (R \Rightarrow Q) \equiv (\neg P \vee Q) \vee (\neg R \vee Q) \equiv \neg P \vee \neg R \vee Q \vee Q$$

$$\equiv \neg P \vee \neg R \vee Q \equiv \neg(P \wedge R) \vee Q$$

$$\equiv (P \wedge R) \Rightarrow Q$$

7. (12pts) Consider the statement: if x is irrational, then $4 + \frac{1}{x}$ is irrational.

a) State the converse and prove or disprove it:

If $4+\frac{1}{x}$ is irrational, then x is irrational.

True: we prove the contrapositive: if x is rational, then $4+\frac{1}{x}$ is rational.
This follows from closure of $\mathbb{Q}$ under addition and division.

b) State the contrapositive and prove or disprove it:

If $4+\frac{1}{x}$ is rational, then x is rational.

True: Suppose $4+\frac{1}{x} = g$, where $g \in \mathbb{Q}$. Then $\frac{1}{x} = g-4$

so $x = \frac{1}{g-4}$, which is a rational number, due to closure of $\mathbb{Q}$ under subtraction and division.

8. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\forall y \in \mathbf{R})(y^2 > x)$$

a) If $x = -3$, is the statement true?
b) If $x = 4$, is the statement true?
c) Find the truth set (the x's) of the above statement.

a) $(\forall y \in \mathbb{R})(y^2 > -3)$ true, since $y^2 \geq 0 > -3$.

b) $(\forall y \in \mathbb{R})(y^2 > 4)$ False since for $y=1$, $y^2 = 1 \not> 4$

c) Truth set $= (-\infty, 0]$.
If $x \leq 0$, then for every $y \in \mathbb{R}$, $y^2 \geq 0 > x$. Thus, $(-\infty, 0] \subseteq$ truth set.
If $x > 0$, then $\sqrt{x}$ is defined, and $\sqrt{x} > 0$. Take $y = \frac{\sqrt{x}}{2}$. Then $y^2 = \frac{x}{4} \not> x$
so $x \notin$ truth set. Thus $(-\infty, 0] =$ truth set.

9. (14pts) Prove using induction: for every natural number n,

$$\frac{1}{2}+1+\frac{3}{2}+2+\frac{5}{2}+3+\cdots+\frac{n}{2}=\frac{n(n+1)}{4}$$

Base: $n=1$ $\quad \frac{1}{2} \stackrel{?}{=} \frac{1\cdot(1+1)}{4}$ yes.

Induction step: suppose statement is true for $n=k$:

$$\frac{1}{2}+1+\frac{3}{2}+\dots+\frac{k}{2}=\frac{k(k+1)}{4} \quad \Big| + \frac{k+1}{2}$$

$$\frac{1}{2}+1+\frac{3}{2}+\dots+\frac{k}{2}+\frac{k+1}{2}=\frac{k(k+1)}{4}+\frac{k+1}{2}=\frac{k(k+1)+2(k+1)}{4}$$

So $\frac{1}{2}+1+\frac{3}{2}+\dots+\frac{k+1}{2}=\frac{(k+1)(k+2)}{4}$, which is the claim for $n=k+1$

10. (12pts) Prove that for all real numbers x and y, $x^2+9y^2 \geq 6xy$.

Investigation

$x^2+9y^2 \geq 6xy$

$x^2-6xy+9y^2 \geq 0$

$(x-3y)^2 \geq 0$

true

Proof: for any $x, y \in \mathbb{R}$

$(x-3y)^2 \geq 0$

$x^2-2\cdot x\cdot 3y+(3y)^2 \geq 0 \quad |+6xy$

$x^2+9y^2 \geq 6xy$

11. (16pts) Consider the statement: for all $a, b \in \mathbf{Z}$, $3 \mid ab$ if and only if $3 \mid a$ or $3 \mid b$.
a) Write the statement as a conjunction of two conditional statements.
b) Prove each of the conditional statements.

a) For all $a, b \in \mathbb{Z}$, if $3 \mid ab$, then $3 \mid a$ or $3 \mid b$ (1)

and if $3 \mid a$ or $3 \mid b$, then $3 \mid ab$. (2)

b) Let $a, b \in \mathbb{Z}$. We assemble a table of multiplication (mod 3)

$b \equiv$ \ $a \equiv$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$ab \equiv$ (mod 3)

From table, we see:

If $a \equiv 0 \pmod 3$ or $b \equiv 0 \pmod 3$, then $ab \equiv 0 \pmod 3$, proving statement (2)

If $ab \equiv 0 \pmod 3$, then either $a \equiv 0 \pmod 3$ or $b \equiv 0 \pmod 3$, proving statement 1.

12. (14pts) Use the statement in problem 11 to show $\sqrt{15}$ is irrational. (Note that a square is a product of two numbers.)

Suppose $\sqrt{15}$ is rational, $\sqrt{15} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$, p, q have no common factors. Then $15 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}$

so $15q^2 = p^2$

$3(5q^2) = p^2$

which implies $3 \mid p^2$. By problem 11, we get that $3 \mid p$, so $p = 3k$ for some $k \in \mathbb{Z}$. Then $3(5q^2) = (3k)^2$

$3(5q^2) = 9k^2$

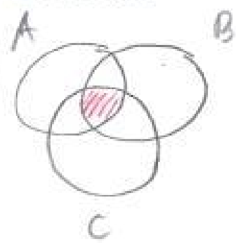
$5q^2 = 3k^2$, which implies $3 \mid 5q^2$.

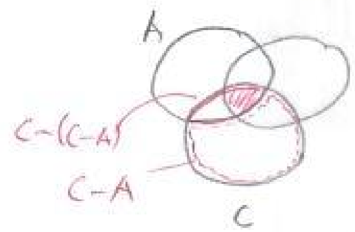
By problem 11, this implies $3 \mid 5$ or $3 \mid q^2$. Since $3 \nmid 5$, we must have $3 \mid q^2$, and thus $3 \mid q$. So $3 \mid p$ and $3 \mid q$, contradicting assumption they have no common factors.

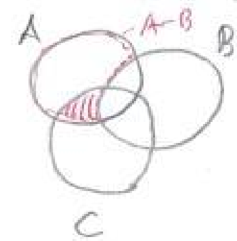
13. (12pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the three sets, two are equal. Use set algebra to show they are equal.

$A \cap B \cap C$ $\qquad\qquad$ $(C - (C - A)) \cap B$ $\qquad\qquad$ $(A - B) \cap C$

a)







b) $\left(C-(C-A)\right)\cap B = \left(C\cap (C\cap A^c)^c\right)\cap B = \left(C\cap (C^c \cup A)\right)\cap B$

$= \left(\underbrace{(C\cap C^c)}_{\emptyset}\cup(C\cap A)\right)\cap B = (C\cap A)\cap B$

$= A\cap B\cap C$

14. (16pts) Let $\mathbf{Z}_5 = \{0,1,2,3,4\}$, and let $f : \mathbf{Z} \to \mathbf{Z}_5$, $f(x) = x^2 + 2x \pmod 5$.
a) Write the table of values of f for $x = 0, 1, 2, 3, 4$.
b) Is f injective? Justify.
c) Is f surjective? Justify.
d) Note the domain is $\mathbf{Z}$ (not $\mathbf{Z}_5$). Determine the set of preimages of 4. List at least three elements of this set and describe the set. The table from a) tells you everything you need to know.

a)

x	x^2+2x	$x^2+2x \pmod 5$
0	0	0
1	3	3
2	8	3
3	15	0
4	24	4

b) f is not injective: $0 \neq 3$, but $f(0) = f(3) = 0$

c) f is not surjective: $2 \notin$ range f.

For every x, $x \equiv r \pmod 5$, $r = 0,1,2,3,4$.

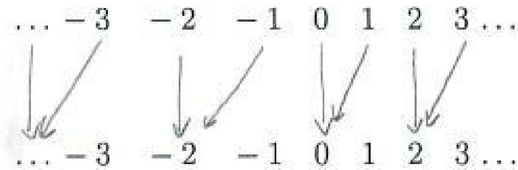
Then $x^2+2x \equiv r^2+2r \pmod 5$ so $f(x) \equiv f(r) \pmod 5$

so $f(x) = f(r)$. Since $f(r) = 0,3,4$, we see that

for every x, $f(x) \neq 2$.

d) Let $f(x) = 4$, for some $x \in \mathbb{Z}$. Then $x \equiv r \pmod 5$, $r = 0,1,2,3,4$, and $f(x) = f(r) \pmod 4$, which gives $f(r) = 4$. The only r for which this is true is $r = 4$, thus $x \equiv 4 \pmod 5$. Set of preimages of 4 is $\{-6, -1, 4, 9, 14, \dots, \}$

15. (5pts) Draw an arrow diagram between the provided two copies of $\mathbf{Z}$ that illustrates a function $f : \mathbf{Z} \to \mathbf{Z}$ that is is not a surjection and the set of preimages of every element in the codomain has either two or zero elements (pattern needs to be obvious).



16. (12pts) Let A, B be subsets of a universal set U. Prove that $A = B$ if and only if $A \cup B = A \cap B$.

$\Rightarrow$) Suppose $A = B$. Then $A \cup B = A \cup A = A = A \cap A = A \cap B$

$\Leftarrow$) Suppose $A \cup B = A \cap B$, and let $x \in A$. Since $x \in A$, $x \in A$ or $x \in B$ so $x \in A \cup B$, which implies $x \in A \cap B$, that is $x \in A$ and $x \in B$.

In particular $x \in B$, so we have proven $A \subseteq B$.

$B \subseteq A$ is shown the same way, so we get $A = B$.

Bonus. (10pts) Let $0 \le a_0, a_1, \ldots, a_n \le 9$ be integers.

a) Use (mod 3) calculus to show that

$$a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \equiv a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0 \pmod 3$$

b) Use a) to show that a natural number is divisible by 3 if and only if the sum of its digits is divisible by 3.

a) $10 \equiv 1 \pmod 3$ so $10^k \equiv 1^k \pmod 3$, that is $10^k \equiv 1 \pmod 3$. Then

$a_n 10^n \equiv a_n \pmod 3$

$a_{n-1} \cdot 10^{n-1} \equiv a_{n-1} \pmod 3$

$a_2 \cdot 10^2 \equiv a_2 \pmod 3$

$a_1 \cdot 10 \equiv a_1 \pmod 3$

$a_0 \equiv a_0 \pmod 3$

Adding the equations, we get

$a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_2 10^2 + a_1 \cdot 10 + a_0 \equiv a_n + \cdots + a_0 \pmod 3$

b) Let $m \in \mathbb{N}$, it can be written as $m = a_n 10^n + \cdots + a_1 10 + a_0$ where $a_n, \ldots, a_0$ are the digits of m. Part a says $3 | m \Leftrightarrow m \equiv 0 \pmod 3 \Leftrightarrow a_n + a_{n-1} + \cdots + a_1 + a_0 \equiv 0 \pmod 3$

$\Leftrightarrow$ sum of digits of m is divisible by 3.