Mathematical Reasoning — Exam 3	
MAT 312, Fall 2020 — D. Ivanšić	

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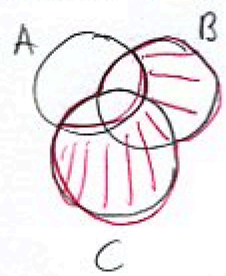
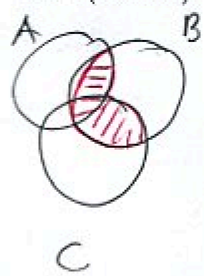
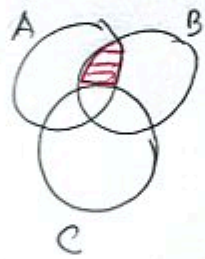
Show all your work!

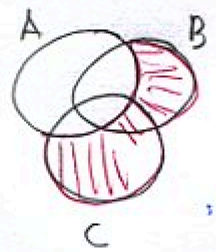
1. (14pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the four sets, two are equal. Use set algebra to show they are equal.

$(A \cup B \cup C) - A$ $\quad$ $B \cap (A \cup C)$ $\quad$ $A \cap (B - C)$ $\quad$ $(B - A) \cup (C - A)$





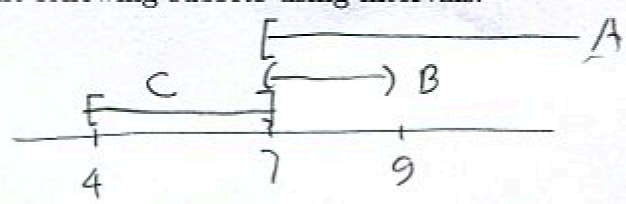




First and fourth appear same

$$(A \cup B \cup C) - A = (A \cup B \cup C) \cap A^c = (A \cap A^c) \cup (B \cap A^c) \cup (C \cap A^c)$$
$$= \phi \cup (B - A) \cup (C - A)$$
$$= (B - A) \cup (C - A)$$

2. (12pts) Let U be the set of real numbers. Consider the intervals of real numbers $A = [7, \infty)$, $B = (7, 9)$, $C = [4, 7]$ and write the following subsets using intervals.



$A \cap C =$ $\{7\}$

$A \cup C =$ $[4, \infty)$

$A^c =$ $(-\infty, 7)$

$A - B =$ $\{7\} \cup [9, \infty)$

$(B \cup C) - A =$ $[4, 7)$

$(C \cup B^c) \cap A =$ $\{7\} \cup [9, \infty)$

3. (14pts) Let $A = \{k \in \mathbf{Z} \mid k \equiv 1 \pmod 3\}$ and $B = \{k \in \mathbf{Z} \mid k \equiv -1 \pmod 3\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $A \cap B = \emptyset$? Prove or disprove.
c) Is $A \cup B = \mathbf{Z}$? Prove or disprove.

$k \equiv 2 \pmod 3$

$A = \{\ldots, -5, -2, 1, 4, 7, \ldots\}$ $B = \{-4, -1, 2, 5, 8, \ldots\}$

a) No. $1 \in A$ but $1 \notin B$ $(3 \nmid 1-(-1))$

b) Yes. Suppose $x \in A \cap B$. Then $x \equiv 1 \pmod 3$ and $x \equiv -1 \pmod 3$
Subtracting the equations gives $0 \equiv 2 \pmod 3$, which is false, giving a contradiction.

c) No. $0 \in \mathbb{Z}$, but $0 \not\equiv 1 \pmod 3$ and $0 \not\equiv -1 \pmod 3$.
so $0 \notin A \cup B$

4. (16pts) Let $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$, and let $f : \mathbf{Z} \to \mathbf{Z}_5$, $f(x) = x^2 + 2x \pmod 5$.
a) Write the table of values of f for $x = 0, 1, 2, 3, 4$.
b) Is f injective? Justify.
c) Is f surjective? Justify.
d) Note the domain is $\mathbf{Z}$ (not $\mathbf{Z}_5$). Determine the set of preimages of 4. List at least three elements of this set and describe the set. The table from a) tells you everything you need to know.

a)

x	$x^2+2x \pmod 5$
0	0
1	3
2	3
3	0
4	4

b) f is not injective, since $1 \neq 2$, but $f(1) = f(2)$

c) f is not surjective: there exists no x s.t. $f(x) = 1$.
This is because for an $x \in \mathbb{Z}$, $x \equiv r \pmod 5$, $r = 0, 1, 2, 3, 4$
so $x^2 + 2x \equiv r^2 + 2r \pmod 5$
$x^2 + 2x \equiv 0, 3, 4 \pmod 5$ for all $x \in \mathbb{Z}$.

d) Let $x \in \mathbb{Z}$ and suppose $f(x) = 4$. If $x \equiv r \pmod 5$
then $f(x) \equiv f(r) \pmod 5$. We get $f(r) \equiv 4 \pmod 5$
only if $r = 4$, thus $x \equiv 4 \pmod 5$
Preimage of 4 is $\{\ldots, -5, -1, 4, 9, 13, \ldots\}$

5. (12pts) Let $f, g : \mathbf{R} \to \mathbf{R}$, $f(x) = \sqrt{x^2}$, $g(x) = x$.
a) The formulas for f and g are different. Are the functions f and g equal?
b) What is the set of preimages of 3 under f?
c) If f and g are not equal, can you alter the domain or codomain so they are?

a) No, because $\left.\begin{array}{l} f(-2) = 2 \\ g(-2) = -2 \end{array}\right\}$ values are not equal

b) $\sqrt{x^2} = 3$ Preimages of $3 = \{-3, 3\}$

$|x| = 3$

$x = \{-3, 3\}$

c) If we restrict domain to $[0, \infty)$, then

$f(x) = \sqrt{x^2} = |x| \underset{\uparrow}{=} x = g(x)$

since $x \geq 0$.

6. (10pts) Let $f(x) = \dfrac{3x}{x-2}$ and assume the codomain is $\mathbf{R}$.
a) What subset of real numbers is the natural domain for this function?
b) What is the range of this function? Justify your answer.

a) Domain $= \{x \mid x \neq 2\}$

(Can't have $x - 2 = 0$, $x = 2$)

b) $y = \dfrac{3x}{x-2}$ $x = \dfrac{2y}{y-3}$ Range $= \{y \mid y \neq 3\}$

$y(x-2) = 3x$

$yx - 2y = 3x$

$yx - 3x = 2y$

$x(y-3) = 2y \leftarrow$ has sol. if $y \neq 3$

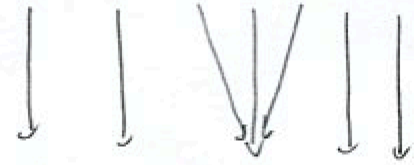
(For $y = 3$ get $0 = 6$)

no sol.

7. (10pts) Draw arrow diagrams between two copies of $\mathbf{Z}$ below that illustrate a function $f : \mathbf{Z} \to \mathbf{Z}$ that is:

a) neither a surjection, nor an injection

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$



$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

b) a bijection that is not the identity

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

8. (12pts) Let A, B be subsets of a universal set U. Prove that $A = B$ if and only if $A \cup B = A \cap B$.

$\Rightarrow$) Suppose $A=B$. Then $A\cup B = A\cup A = A = A\cap A = A\cap B$

$\Leftarrow$) Suppose $A\cup B = A\cap B$.

We show $A\subseteq B$: Let $x\in A$. Then $x\in A\cup B$, so $x\in A\cap B$, which means $x\in A$ and $x\in B$, so, in particular $x\in B$. Thus $A\subseteq B$

The inclusion $B\subseteq A$ goes the same way.

Bonus. (10pts) Let $f : [3, \infty) \to \mathbf{R}$, $f(x) = x^2 - 6x + 8$. The graph of f easily shows that f is injective. Prove injectivity algebraically. *(Hint: difference of squares.)*

Let $f(x_1) = f(x_2)$, $x_1, x_2 \in [3,\infty)$

$x_1^2 - 6x_1 + 8 = x_2^2 - 6x_2 + 8$

$x_1^2 - x_2^2 - 6x_1 + 6x_2 = 0$

$(x_1 - x_2)(x_1 + x_2) - 6(x_1 - x_2) = 0$

$(x_1 - x_2)(x_1 + x_2 - 6) = 0$

It follows that

$x_1 - x_2 = 0$, so $x_1 = x_2$

or $x_1 + x_2 - 6 = 0$

$x_2 = 6 - x_1$

Since $x_1 \geq 3$

$-x_1 \leq -3$

$6 - x_1 \leq 3$

since $x_2 \geq 3$ we get

$3 \leq 6 - x_1 \leq 3$

so $6 - x_1 = 3$, thus

$x_1 = 3, x_2 = 3$.

In either case, we get $x_1 = x_2$