Mathematical Reasoning — Exam 2
MAT 312, Fall 2020 — D. Ivanšić

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Show all your work!

1. (14pts) Prove using induction: for every integer $n \geq 1$,
$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Base step: $n=1$ $\quad 1^2 = \frac{1\cdot 2\cdot 3}{6}$, true

Suppose equation is true for k: $1^2+2^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6}$

Then $1^2+2^2+\dots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = (k+1)\left(\frac{k(2k+1)}{6} + k+1\right)$

$= (k+1)\left(\frac{2k^2+k}{6} + \frac{6(k+1)}{6}\right) = (k+1)\frac{2k^2+7k+6}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

$= \frac{(k+1)(k+2)(2(k+1)+1)}{6}$, the statement for $k+1$.

2. (14pts) Prove: for any integer n, the number $\frac{2}{3}n^3 + \frac{4}{3}n$ is an integer.

By induction:

$n \geq 0$, for $n=0$ obvious,

Suppose $\frac{2}{3}k^3 + \frac{4}{3}k = q \in \mathbb{Z}$

$\frac{2}{3}(k+1)^3 + \frac{4}{3}(k+1)$

$= \frac{2}{3}(k^3+3k^2+3k+1) + \frac{4}{3}(k+1)$

$= \frac{2}{3}k^3 + \frac{4}{3}k + \frac{2}{3}\cdot 3(k^2+k) + \frac{2}{3} + \frac{4}{3}$

$= q + 2(k^2+k) + 2$ which is an integer.

If $n<0$, $\frac{2}{3}(-n)^3 + \frac{4}{3}(-n)$ is an integer, so $-\frac{2}{3}n^3 - \frac{4}{3}n \in \mathbb{Z}$ so its opposite $\frac{2}{3}n^3 + \frac{4}{3}n$ is an integer.

Using mod calculus:

$\frac{2}{3}n^3 + \frac{4}{3}n = \frac{2n^3+4n}{3}$ statement is equivalent to $2n^3+4n$ being a multiple of 3.

$n \equiv$	$2n^3+4n \equiv \pmod 3$
0	0
1	$2+4=6\equiv 0$
2	$16+8=24\equiv 0$

From table, we see that if $n \equiv r \pmod 3$ $r=0,1,2$, then $2n^3+4n \equiv 0 \pmod 3$

so $3 \mid 2n^3+4n$

3. (16pts) We know that $\sqrt{2}$ is irrational. Are the following statements true? Justify with a counterexample or a proof.

a) There exists a real number x such that $\dfrac{1}{x+\sqrt{2}}$ is rational.

b) If x is rational, then $\dfrac{1}{x+\sqrt{2}}$ is irrational.

a) True. Let $x = 1-\sqrt{2}$, then $\dfrac{1}{1-\sqrt{2}+\sqrt{2}} = 1 \in \mathbb{Q}$

b) Suppose x is rational and $\dfrac{1}{x+\sqrt{2}}$ is rational. Then

$\dfrac{1}{x+\sqrt{2}} = q$ for some $q \in \mathbb{Q}$

$x+\sqrt{2} = \dfrac{1}{q}$ $\qquad$ $\dfrac{1}{q} - x$ is a rational number due to closure of $\mathbb{Q}$ under division and subtraction.

$\sqrt{2} = \dfrac{1}{q} - x$

4. (18pts) Consider the statement: for all $a, b \in \mathbf{Z}$, $5 \mid a^2+2b^2$ if and only if $5 \mid a$ and $5 \mid b$.

a) Write the statement as a conjunction of two conditional statements.

b) Determine whether each of the conditional statements is true, and write a proof, if so.

c) Is the original statement true?

a) (If $5 \mid a^2+2b^2$, then $5 \mid a$ and $5 \mid b$) and

(If $5 \mid a$ and $5 \mid b$, then $5 \mid a^2+2b^2$)

$b\equiv$ \ $a\equiv$	0	1	2	3	4
0	0	1	4	4	1
1	2	3	1	1	3
2	3	4	2	2	4
3	3	4	2	2	4
4	2	3	1	1	3

$a^2+2b^2 \equiv \square \pmod 5$

b) From table we see:

1) If $5 \mid a$ and $5 \mid b$, then $a, b \equiv 0 \pmod 5$ so $a^2+2b^2 \equiv 0 \pmod 5$, so $5 \mid a^2+2b^2$

2) If $5 \mid a^2+2b^2$, then $a^2+2b^2 \equiv 0 \pmod 5$. This happens only when $a \equiv 0 \pmod 5$ and $b \equiv 0 \pmod 5$.

c) The biconditional statement is true because both conditional statements are true

5. (14pts) We have shown a similar statement on homework: for every integer n, if $7 \mid n^2$, then $7 \mid n$. Use this proposition to show that $\sqrt{7}$ is irrational.

Suppose $\sqrt{7}$ is rational. Then there exist $m, n \in \mathbb{Z}$ such that $\frac{m}{n} = \sqrt{7}$ and $\frac{m}{n}$ is reduced. Then $\frac{m^2}{n^2} = 7$ so $m^2 = 7n^2$.

This means that $7 \mid m^2$, so $7 \mid m$, and $m = 7g$ for some $g \in \mathbb{Z}$.

Then $(7g)^2 = 7n^2$, $49g^2 = 7n^2$, $7g^2 = n^2$ so $7 \mid n^2$ and therefore, $7 \mid n$. But then $7 \mid m$ and $7 \mid n$ contradicts the assumption that $\frac{m}{n}$ is reduced.

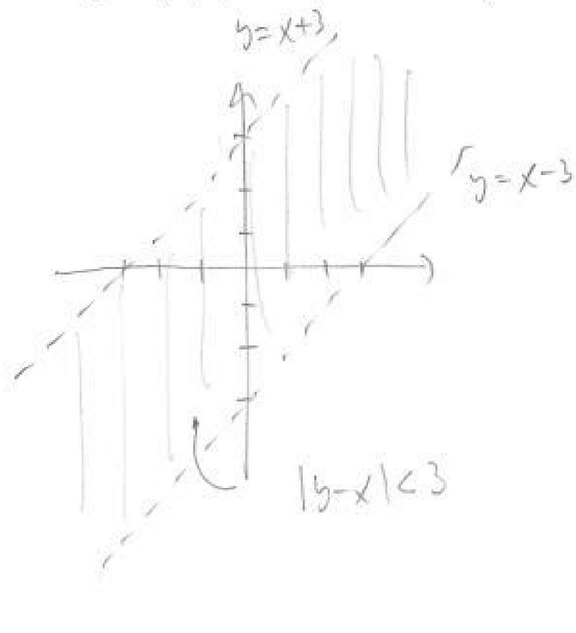
6. (10pts) Sketch all points (x, y) in the plane that satisfy $|y - x| < 3$. (Hint: what inequalities without absolute value is the inequality $|u| < a$ equivalent to?)

$|u| < a \iff -a < u < a$

$|y - x| < 3$

$\iff -3 < y - x < 3$

$x - 3 < y < x + 3$



7. (14pts) Prove both statements for all real numbers x (one is easy):

a) if $x < -1$, then $x + \frac{1}{x+1} < 0$; b) if $x > -1$, then $x + \frac{1}{x+1} \geq 1$.

a) If $x < -1$, then

so $x < 0$ and $x + 1 < 0$

so $\frac{1}{x+1} < 0$, thus

$x + \frac{1}{x+1} < 0$

b) $x + \frac{1}{x+1} \geq 1 \quad | \cdot x+1$

$x^2 + x + 1 \geq x + 1$

$x^2 \geq 0$

true

Investigation

Proof. Let $x > -1$. Then $x + 1 > 0$.

For any x, $x^2 \geq 0$

$x^2 + x + 1 \geq x + 1 \quad | \div (x+1) > 0$

$\frac{x^2 + x}{x+1} + \frac{1}{x+1} \geq 1$

$x + \frac{1}{x+1} \geq 1$

Bonus. (10pts) Let $0 \leq a_0, a_1, \ldots, a_n \leq 9$ be integers.

a) Use (mod 3) calculus to show that

$a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \equiv a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0 \pmod 3$

b) Use a) to show that a natural number is divisible by 3 if and only if the sum of its digits is divisible by 3.

a) Since $10 \equiv 1 \pmod 3$ we get $10^k \equiv 1^k \equiv 1 \pmod 3$

Multiplying $10^k \equiv 1 \pmod 3$ by a_k we get.

$a_0 \equiv a_0 \pmod 3$

$10 a_1 \equiv a_1 \pmod 3$

$10^2 a_2 \equiv a_2 \pmod 3$

$\vdots$

$10^n a_n \equiv a_n \pmod 3$

Adding these equations gives

$a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1 10 + a_0 \equiv a_n + a_{n-1} + \ldots + a_1 + a_0 \pmod 3$

b) If m is a natural number with digits $a_n a_{n-1} \ldots a_1 a_0$, then $m = a_n 10^n + \ldots + a_1 10 + a_0$

Statement a) says $m \equiv a_n + \ldots + a_1 + a_0 \pmod 3$

so $3 | m \Leftrightarrow m \equiv 0 \pmod 3 \Leftrightarrow a_n + \ldots + a_1 + a_0 \equiv 0 \pmod 3$

$\Leftrightarrow 3 |$ sum of digits of m.