Mathematical Reasoning — Exam 1 MAT 312, Fall 2020 — D. Ivanšić

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Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) The sky is up and the ground is down.

True and true is true

2. (2pts) For every $x \in \mathbf{R}$, $x^3 > 0$ or $\underbrace{2 + 5 = 7}_{\text{true}}$.

true or anything is true, so entire statement is true

3. (3pts) If $f'(x) = x^2$, then $f(x) = \dfrac{x^3}{3}$.

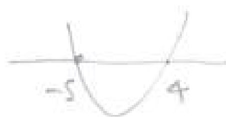
False, Counterexample: $\frac{x^3}{3} - 2$.

4. (4pts) (universal set=$\mathbf{Z}$) $x^2 + x < 20$

Truth set: $\{-4, -3, -2, -1, 0, 1, 2, 3\}$

$x^2 + x - 20 = 0$

$(x+5)(x-4) = 0$



5. (3pts) (universal set=$\mathbf{R}$) $x^4 + x^2 = 0$

Truth set $= \{0\}$

$x^4 + x^2 = 0$

$x^2(x^2 + 1) = 0$

$x^2 = 0$ or $x^2 + 1 = 0$

$x = 0$ no sol.

Negate the following statements.

6. (3pts) I am a poor boy and I don't need sympathy.

I am not a poor boy "or" I need sympathy

7. (3pts) If they send me away, then they teach me how to be sensible.

They send me away and they don't teach me how to be sensible.

8. (8pts) Use a truth table to prove that $(P \vee Q) \wedge \neg Q \equiv \neg(P \Longrightarrow Q)$. (Use however many columns you need.)

P	Q	$\neg Q$	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$			
T	T	F	T	F	T	F			
T	F	T	T	T	F	T			
F	T	F	T	F	T	F			
F	F	T	F	F	T	F			

9. (12pts) Use previously proven logical equivalences to prove the equivalence $P \Longrightarrow (Q \wedge R) \equiv (P \Longrightarrow Q) \wedge (P \Longrightarrow R)$. Do not use a truth table.

$P \Rightarrow (Q \wedge R) \equiv \neg P \vee (Q \wedge R) \equiv (\neg P \vee Q) \wedge (\neg P \wedge R) \equiv (P \Rightarrow Q) \wedge (P \Rightarrow R)$

10. (4pts) Write the converse and contrapositive of the statement: if $x^3 - x - 7 > 0$, then $x > 0$.

Converse: If $x > 0$, then $x^3 - x - 7 > 0$

Contrapositive: If $x \leq 0$, then $x^3 - x - 7 \leq 0$.

11. (8pts) Suppose the following statements are true:
If the wedding is in China, the bride wears red. $P \Rightarrow Q$ is true
The wedding is in China or the bride wears red. $P \vee Q$ is true

Determine truth value of the following statement and justify: the bride wears red. Q

Since $P \Rightarrow Q$ and $P \vee Q$ are true so is $(P \Rightarrow Q) \wedge (P \vee Q) \equiv (\neg P \vee Q) \wedge (P \vee Q)$

$\equiv (\neg P \wedge P) \vee Q \equiv F \vee Q \equiv Q$, so Q is true

12. (4pts) Use set builder notation to write the set $\{4, 8, 16, 32, 64, \ldots\}$.

$\{x \in \mathbb{N} \mid x = 2^n \text{ for some } n \in \mathbb{N}, n \geq 2\} = \{2^n \mid n \in \mathbb{N} \text{ and } n \geq 2\}$

13. (10pts) A function $f : \mathbf{R} \to \mathbf{R}$ is *bounded above* if there exists an $M \in \mathbf{R}$ such that for every $x \in \mathbf{R}$, $f(x) \leq M$.

a) Write the definition using symbols for quantifiers.

b) Negate the definition using symbols for quantifiers.

c) Finish the sentence: "A function $f : \mathbf{R} \to \mathbf{R}$ is not bounded above if ..."

a) $(\exists M \in \mathbb{R})(\forall x \in \mathbb{R})(f(x) \leq M)$

b) $(\forall M \in \mathbb{R})(\exists x \in \mathbb{R})(f(x) > M)$

A function $f : \mathbb{R} \to \mathbb{R}$ is not bounded above if for every real number M, there is a real number x such that $f(x) > M$.

14. (7pts) Prove: if m is an even integer, and n is an odd integer, then $m + n$ is an odd integer.

If m is even and n is odd, then $m = 2k$ and $n = 2l+1$ for some integers k, l. Then $m+n = 2k + 2l + 1 = 2(k+l)+1$. Since $k+l$ is an integer, $m+n$ is odd.

15. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\exists y \in \mathbf{R})(x - y^2 = 3)$$

a) If $x = 0$, is the statement true?

b) If $x = 5$, is the statement true?

c) Find the truth set (the x's) of the above statement.

a) $x=0$ $(\exists y \in \mathbb{R})(-y^2 = 3)$ false, since $y^2 \geq 0$ so $-y^2 \leq 0$

b) $x=5$ $(\exists y \in \mathbb{R})(5 - y^2 = 3)$ $5 - y^2 = 3$, $y^2 = 2$, $y = \pm\sqrt{2}$ true

c) For which x does $x - y^2 = 3$ have a solution for y?

$x - 3 = y^2$ $\quad\sqrt{x-3}$ is a real number when $x - 3 \geq 0$, i.e. $x \geq 3$

$y = \pm\sqrt{x-3}$ $\quad$ Truth set: $[3, \infty)$

16. (15pts) We will call an integer n type-0, type-1 or type-2 integer if it can be written in the form $n = 3k$, $n = 3k+1$ or $n = 3k+2$, respectively, for some integer k. Prove that if m is a type-1 integer and n is a type-2 integer, then $m^2 - n^2 + n$ is a type-2 integer. Start with a know-show table if you find it helpful.

Suppose m is type-1 and n is type 2. Then there exist integers k, l such that $m = 3k+1$ and $l = 3l+2$. Then

$m^2 - n^2 + n = (3k+1)^2 - (3l+2)^2 + 3l+2 = 9k^2 + 6k+1 - (9l^2 + 12l+4) + 3l+2$

$= 9k^2 + 6k - 9l^2 - 9l - 1$

$= 9k^2 + 6k - 9l^2 - 9l - 3 + 2$

$= 3(3k^2 + 2k - 3l^2 - 3l - 1) + 2$

Since $3k^2 + 2k - 3l^2 - 3l - 1$ is an integer, $m^2 - n^2 + n$ is type 2

Bonus. (10pts) Show by that quantifiers are not "commutative" by showing that one of the statements below is true and the other is false (justify).

$(\exists y \in \mathbf{R})(\forall x \in \mathbf{R})(x > y)$

False. For any y, there is an x such that $x \leq y$. (for example, $x = y - 1$)

$(\forall x \in \mathbf{R})(\exists y \in \mathbf{R})(x > y)$

True. Given x, $y = x - 1$ satisfies $x > y$.