Calculus 2 — Exam 0
MAT 308, Fall 2020 — D. Ivanšić

Name: ______________________________

Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\dfrac{d}{dx}\left(8x^3 - c^6 + \sqrt[6]{x^{11}} - \dfrac{2}{x^3}\right) =$

2. (6pts) $\dfrac{d}{dt}\,(t^2-1)^3\,(t^2+1)^3 =$

3. (8pts) $\dfrac{d}{dw}\,\dfrac{\sqrt[3]{w} + \dfrac{1}{\sqrt[3]{w}}}{w+1} =$

4. (4pts) $\dfrac{d}{d\theta}\,\ln(\theta\cos\theta) =$

5. (7pts) (This is a known derivative, your job is to verify it here.)

$\dfrac{d}{dx}\,\ln|x + \sqrt{x^2+1}| =$

6. (6pts) $\dfrac{d}{dx}\,e^{\sqrt{\arctan x}} =$

7. (5pts) Let $f(x) = xe^{2x}$. Take the first four derivatives of f, and try to spot the pattern. What is $f^{(36)}(x)$, the 36th derivative of f? How about $f^{(n)}(x)$?

Find the following limits. Use L'Hospital's rule if needed.

8. (2pts) $\lim\limits_{x \to \infty} \dfrac{1}{\sqrt{x}} =$

9. (6pts) $\lim\limits_{x \to \infty} \dfrac{x^2 - 3x + 1}{x^3 - 3x^2 + 6x + 5} =$

10. (8pts) $\lim\limits_{x \to 0} (1 + x^2)^{\frac{1}{x}} =$

Find the following antiderivatives.

11. (7pts) $\displaystyle\int 3x^5 - \frac{5}{1+x^2} + \frac{2}{\sqrt[4]{x^9}} + e^{\pi}\,dx =$

12. (3pts) $\displaystyle\int (4x+11)^8\,dx =$

13. (7pts) $\displaystyle\int \frac{\sqrt{x} - \dfrac{1}{\sqrt{x}}}{x^2}\,dx =$

Use the substitution rule in the following integrals:

14. (7pts) $\displaystyle\int (x+3)\cos(x^2+6x-7)\,dx =$

15. (10pts) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos x\, 3^{\sin x}\,dx =$

16. (8pts) Consider the integral $\int_3^7 x^2 - 4x - 5\,dx$.
a) Draw a picture to explain the meaning of the integral.
b) Use the picture to estimate whether the integral is positive or negative.
c) Evaluate the integral to verify your finding in b).

Bonus. (10pts) The rear inside cover of our book claims that

$$\int \frac{\sqrt{a^2 - x^2}}{x^2}\,dx = -\frac{\sqrt{a^2 - x^2}}{x} - \arcsin\frac{x}{a} + C$$

Verify this formula by differentiating.

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Show all your work!

Find the following integrals:

1. (7pts) $\displaystyle\int xe^{3x}\,dx =$

2. (7pts) $\displaystyle\int \sin^2 x\,dx =$

Determine whether the following improper integral converges, and, if so, evaluate it. (Calculate directly, comparison would be hard.)

3. (14pts) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx =$

Use trigonometric substitution to evaluate the following integrals. Don't forget to return to the original variable where appropriate.

4. (14pts) $\displaystyle\int \frac{x^3}{\sqrt{x^2-1}}\,dx =$

5. (14pts) $\displaystyle\int_0^{\frac{3}{2}} \frac{1}{(9-x^2)^{\frac{3}{2}}}\,dx =$

Use the method of partial fractions to find the following integrals.

6. (14pts) $\displaystyle\int \frac{-x^2 - 3x + 2}{(x+1)(x^2+1)}\, dx =$

7. (10pts) Use comparison to determine whether the improper integral $\displaystyle\int_1^{\infty} \frac{x^2}{x^4+7}\, dx$ converges.

8. (20pts) Suppose we wanted to approximate the number $\ln 4$. We could do it by approximating the integral $\int_1^4 \frac{1}{x}\,dx = \ln 4$, which uses only the four algebraic operations.
a) Write the expression you would use to calculate T_6, the trapezoid rule with 6 subintervals. All the terms need to be explicitly written, do not use f in the sum.
b) Find the error estimate for T_n in general. You will need the second derivative of $\frac{1}{x}$.
c) Estimate the error for T_6.
d) What should n be in order for T_n to give you an error less than 10^{-4}?

Bonus (10pts) On the interval $[1,3]$, draw a nice big picture of any concave upward function f whose graph is above the x-axis. Then draw the straight-edge shapes whose area is represented by the trapezoid and midpoint approximations T_2 and M_2 for the integral $I = \int_1^3 f(x)\,dx$. Put the numbers I, T_2 and M_2 in increasing order and justify this order precisely with your picture.

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Show all your work!

1. (24pts) The region bounded by the lines $y = x$ and $y = 2x - 1$ and $x = 3$ is rotated around the x-axis.
a) Sketch the solid and a typical cross-sectional washer.
b) Set up the integral for the volume of the solid.
c) Evaluate the integral.

2. (14pts) Consider the region bounded by curves $y = e^x$, $y = e^{-x}$ and $y = \frac{1}{3}$.
a) Sketch the region.
b) Set up the integral that computes its area. Simplify, but do not evaluate the integral.

3. (16pts) Rotate the region bounded by the curves $y = \sqrt{x}$, $y = x - 2$ and the x-axis about the x-axis to get a solid.
a) Sketch the solid and a typical cylindrical shell.
b) Set up the integral for the volume of the solid using the shell method. Simplify, but do not evaluate the integral.

4. (18pts) Consider the arc of the circle $x^2 + y^2 = 9$ that is to the right of the line $x = 1$. Rotate it around around the x-axis to get a spherical cap. Find the surface area of this cap.

5. (12pts) Set up the integral for the length of the curve $y = x^3 - 4x$ from $x = 0$ to $x = 2$. Simplify, but do not evaluate the integral.

6. (16pts) A 15-foot chain weighing 45lbs is lying on the ground. Set up the computation for the work done if one of its ends is hoisted to height 25 feet, pulling the rest of the chain up. Simplify, but do not evaluate any integrals.

Bonus (10pts) Cross sections by parallel vertical planes of a 5-meter long tank are like the region between the curves $y = x^2$ and $y = 8$. (Thus, the tank is 8 meters tall and has a flat top.) Set up the integral for the work needed to fill this tank with water, assuming the water is pumped from 2 meters below the tank. Assume $g = 10$ and water density $= 1000\text{kg/m}^3$. Do not evaluate the integral, but do draw copious pictures!

Calculus 2 — Exam 3
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Show all your work!

Find the limits, if they exist.

1. (6pts) $\lim_{n\to\infty} \frac{n}{1.5^n} =$

2. (6pts) $\lim_{n\to\infty} (2 + (-1)^n) =$

3. (10pts) Find the limit. Use the theorem that rhymes with the insects typically found on cats and dogs.

$$\lim_{n\to\infty} \frac{2^n(\sin(17n) + 1)}{3^n}$$

4. (6pts) Write the series using summation notation:

$$\frac{3}{8} - \frac{7}{16} + \frac{11}{32} - \frac{15}{64} + \cdots =$$

5. (12pts) Justify why the series converges and find its sum.

$$\sum_{n=2}^{\infty} \frac{2 \cdot 3^{2n-1}}{2^{4n+3}} =$$

Determine whether the following series converge and justify your answer.

6. (6pts) $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{n}$

7. (12pts) $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sqrt[n]{5}}{3n^2}$

8. (20pts) Consider the alternating series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n+\sqrt{n}}$.

a) Is the series convergent? Justify.
b) Is the series absolutely convergent? Justify.

Determine whether the following series converge using the root or ratio test.

9. (11pts) $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{22^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

10. (11pts) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 + 3n^2 + 1}{2^{3n+1}}$

Bonus. (10pts) Play this game on a basic calculator: enter any positive number, then keep pressing the $\sqrt{\ }$ key. After a while, the display stabilizes at a number. (In case you have never used a basic calculator, pressing $\sqrt{\ }$ immediately returns the square root of the number.)

a) Use a sequence and a limit to explain what is happening.

b) At which number does the display stabilize?

Calculus 2 — Exam 4
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Show all your work!

Find the intervals of convergence for the series below. Don't forget to check the endpoints.

1. (16pts) $\displaystyle\sum_{n=0}^{\infty} 3^n \cdot \sqrt{n} \cdot (x-4)^n$

2. (10pts) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{(2n)!}\, x^n$

3. (6pts) Use a known power series to find the sum:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n3^n} =$$

4. (8pts) Use a known power series to find the limit.

$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x^3} =$$

5. (14pts) Use geometric series and differentiation to get a power series for $\dfrac{x^2}{(1-x^2)^2}$. State the interval of convergence (no need to check the endpoints).

6. (12pts) Use known power series to show that $\dfrac{d}{dx}\cos x = -\sin x$.

7. (18pts) Let $f(x) = \sqrt[3]{x}$.
a) Find the 2nd Taylor polynomial for f centered at $a = 8$.
b) Use Taylor's formula to get an estimate of the error $|R_2|$ on the interval $[6.5, 9.5]$. Leave your answer as a fraction.

8. (16pts) Use the known power series for $\cos x$ to find the series representing $\int_0^{\frac{1}{2}} \cos\sqrt{x}\, dx$. (Note that $\cos\sqrt{x}$ does not have an antiderivative that is an elementary function.) Give an approximation of the definite integral with accuracy 10^{-3}. Write the approximation as a sum (you do not have to simplify it).

Bonus (10pts) Find a fraction that is the approximation of $\sqrt{5}$ with accuracy 10^{-2}. Start as below and take advantage of the binomial series.

$$\sqrt{5} = \sqrt{4\left(1 + \frac{1}{4}\right)} =$$

Calculus 2 — Exam 5
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Name: ______________________

Show all your work!

1. (12pts) Polar coordinates of two points are given.
a) Sketch the points in the plane.
b) For each point, give two additional polar coordinates, one with a negative r, one with a negative θ.

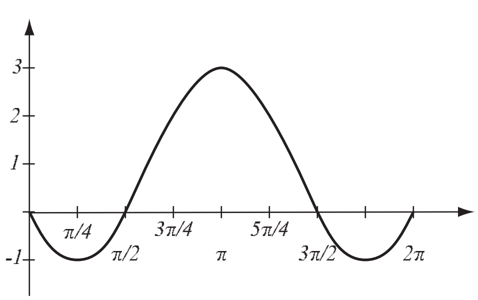
$\left(2, \frac{5\pi}{6}\right)$ $\qquad$ $\left(-4, \frac{3\pi}{7}\right)$

2. (10pts) Convert (a picture may help):
a) $\left(4, -\frac{\pi}{6}\right)$ from polar to rectangular coordinates
b) $(-2\sqrt{2}, -2\sqrt{2})$ from rectangular to polar coordinates

3. (14pts) Find the equation of the tangent line to the parametric curve $x = e^t \cos t$, $y = e^t \sin t$ at the point where $t = 0$.

4. (12pts) A particle moves along the path with parametric equations $x(t) = 3t^2 - 6$, $y(t) = t^2$ for $-2 \leq t \leq 2$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

5. (12pts) The graph of $r = f(\theta)$ is given in cartesian coordinates. Use its intervals of increase and decrease to help you sketch the polar curve $r = f(\theta)$. Indicate which piece of the cartesian graph corresponds to which piece of the polar graph.



6. (12pts) Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, given by parametric equations $x(t) = a\cos t$, $y(t) = b\sin t$, $0 \le t \le 2\pi$, where a and b are constants.

7. (10pts) In celebration of the end of the semester, an empty tub of disinfectant wipes is launched from the roof of Faculty Hall, so that its position is given by $x(t) = 10t$, $y(t) = 20 + 15t - 5t^2$, where length is measured in meters, time in seconds.

a) Splat! When does the tub hit the ground?

b) What horizontal distance did it travel until touchdown?

8. (18pts) A parametric curve is given by $x(t) = t^3 - 12t$, $y(t) = t^2 + 2t - 8$.
a) Find the points on the curve where the tangent line is horizontal or vertical.
b) Where does the curve go as $t \to \infty$ and $t \to -\infty$? (That is, find $\lim_{t \to \pm\infty} x(t)$, $\lim_{t \to \pm\infty} y(t)$.)
c) Plot the points from a) on a coordinate system and use them, along with information from b), or from plotting additional points, to get a graph of the curve. Recall that the curve moves in only one of general directions $\nearrow \nwarrow \swarrow \searrow$ between points from a).

Bonus. (10pts) Find the intervals of concavity for the parametric curve given in problem 8.

Calculus 2 — Final Exam MAT 308, Fall 2020 — D. Ivanšić

Name: ____________________

Show all your work!

Find the following integrals:

1. (6pts) $\displaystyle\int \frac{\ln x}{x^4}\,dx =$

2. (10pts) $\displaystyle\int \cos^4 x \sin^3 x\,dx =$

3. (12pts) Use trigonometric substitution to evaluate the integral.

$$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx =$$

4. (12pts) Consider the improper integral below.
a) Determine whether the improper integral converges, and, if so, evaluate it.
b) Use your answer from a) to decide the convergence of the series below. Which tests are you using?

$$\int_1^\infty \frac{x^2}{1+x^3}\,dx =$$

$$\sum_{n=1}^{\infty} \frac{5n^2}{1+n^3}$$

5. (14pts) Find the interval of convergence of the series. Don't forget to check the endpoints.

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{3^n \cdot \sqrt{n}}$$

6. (10pts) Justify why the series converges and find its sum.

$$\sum_{n=1}^{\infty} \frac{3 \cdot 2^{2n}}{5^{n+1}} =$$

7. (24pts) The region bounded by the curves $y = 9 - x^2$ and $y = -3x + 9$ is rotated around the x-axis.
a) Sketch the solid and a typical cross-sectional washer.
b) Set up the integral for the volume of the solid.
c) On another picture, sketch the solid and a typical cylindrical shell.
d) Set up the integral for the volume of the solid using the shell method.
Simplify, but do not evaluate the integrals.

8. (16pts) Let $f(x) = \cos x$.
a) Find the 3rd Taylor polynomial for f centered at $a = \dfrac{\pi}{2}$.
b) Use Taylor's formula to get an estimate of the error $|R_3|$ on the interval $\left(\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right)$.

9. (10pts) A particle moves along the path with parametric equations $x(t) = 3 + \sin t$, $y(t) = 1 - 2\sin t$, $0 \le t \le 2\pi$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

10. (24pts) The integral $\int_0^1 \cos\sqrt{x}\,dx$ is given. It cannot be found by antidifferentiation, since the antiderivative of $\cos\sqrt{x}$ is not expressible using elementary functions.
a) Write the expression you would use to calculate M_6, the midpoint rule with 6 subintervals. All the terms need to be explicitly written, do not use f in the sum.
b) It is known that $0.3 < y'' < \frac{1}{3}$ on $[0, 1]$: use it to find the error estimate for M_n in general.
c) What should n be in order for M_n to give you an error less than 10^{-4}?
d) Use a known power series for to find a power series for the above integral.
e) How many terms of the power series are needed to estimate the integral to accuracy 10^{-4}? Write the estimate as a sum (you do not have to simplify it).
f) Which method requires less computation to evaluate the integral with accuracy 10^{-4}, midpoint formula or series?

11. (12pts) First draw the graph of $r = 1 - \sin\theta$ in a cartesian θ-r coordinates. Use this graph to draw the polar curve with the same equation.

Bonus (15pts) Find a fraction that is the approximation of $\sqrt[3]{9}$ with accuracy 10^{-3}. Start as below and take advantage of the binomial series.

$$\sqrt[3]{9} = \sqrt[3]{8\left(1 + \frac{1}{8}\right)} =$$