Calculus 2 — Final Exam **MAT 308, Fall 2020 — D. Ivanšić**	**Name:** Saul Ocean *Show all your work!*

Find the following integrals:

1. (6pts) $\int \frac{\ln x}{x^4}\,dx = \begin{bmatrix} u=\ln x & dv = x^{-4}dx \\ du = \frac{1}{x}dx & v = \frac{x^{-3}}{-3} \end{bmatrix} = -\frac{x^{-3}}{3}\ln x - \int -\frac{x^{-3}}{3}\cdot\frac{1}{x}\,dx$

$= -\frac{\ln x}{3x^3} + \int \frac{1}{3}x^{-4}\,dx$

$= -\frac{\ln x}{3x^3} - \frac{1}{9}x^{-3} = -\frac{\ln x}{3x^3} - \frac{1}{9x^3}$

2. (10pts) $\int \cos^4 x \sin^3 x\,dx = \int \cos^4 x \underbrace{\sin^2 x}_{1-\cos^2 x} \sin x\,dx = \begin{bmatrix} u = \cos x \\ du = -\sin x\,dx \end{bmatrix}$ $-du = \sin x\,dx$

$= \int u^4(1-u^2)(-du)$

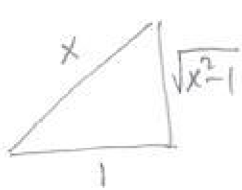
$= \int u^6 - u^4\,du = \frac{u^7}{7} - \frac{u^5}{5} = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$

3. (12pts) Use trigonometric substitution to evaluate the integral.

$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \begin{bmatrix} x = \sec\theta \\ dx = \sec\theta\tan\theta\,d\theta \end{bmatrix} = \int \frac{\sec\theta\tan\theta}{\sec^2\theta\underbrace{\sqrt{\sec^2\theta - 1}}_{\tan^2\theta}}\,d\theta$

$= \int \frac{\cancel{\sec\theta}\,\cancel{\tan\theta}}{\sec^{\cancel{2}}\theta\,\cancel{\tan\theta}}\,d\theta = \int \frac{d\theta}{\sec\theta}$

$= \int \cos\theta\,d\theta = \sin\theta = \frac{\sqrt{x^2-1}}{x} + C$



$\frac{x}{1} = \sec\theta$

$\frac{1}{x} = \cos\theta$

4. (12pts) Consider the improper integral below.
a) Determine whether the improper integral converges, and, if so, evaluate it.
b) Use your answer from a) to decide the convergence of the series below. Which tests are you using?

$$\int_1^\infty \frac{x^2}{1+x^3}\,dx = \lim_{t\to\infty} \int_1^t \frac{x^2}{1+x^3}\,dx = \lim_{t\to\infty} \frac{\ln|1+x^3|}{3}\Big|_1^t = \lim_{t\to\infty} \frac{1}{3}\left(\ln|1+t^3| - \ln 2\right)$$

$$= \frac{1}{3}(\ln\infty - \ln 2) = \frac{1}{3}(\infty - 2) = \infty \quad \text{integral diverges}$$

$$\sum_{n=1}^{\infty} \frac{5n^2}{1+n^3} \qquad \frac{5n^2}{1+n^3} > \frac{n^2}{1+n^3}$$

Since $\int_1^\infty \frac{x^2}{1+x^3}\,dx$ diverges, $\sum \frac{n^2}{1+n^3}$ diverges by integral test. $\sum \frac{5n^2}{1+n^3}$ then diverges by comparison test.

5. (14pts) Find the interval of convergence of the series. Don't forget to check the endpoints.

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{3^n \cdot \sqrt{n}} \qquad \sqrt[n]{\left|\frac{(x-4)^n}{3^n\sqrt{n}}\right|} = \frac{|x-4|}{3\cdot\sqrt[n]{\sqrt{n}}} = \frac{|x-4|}{3\underbrace{\sqrt{\sqrt[n]{n}}}_{\to 1}} \to \frac{|x-4|}{3\cdot 1} = \frac{|x-4|}{3}$$

$$\frac{|x-4|}{3} < 1$$

$$|x-4| < 3$$

$-3 \quad +3$

$1 \quad 4 \quad 7$

When $x=7$, get $\sum_{n=1}^{\infty} \frac{\cancel{3^n}}{\cancel{3^n}\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ p-series, $p<1$, diverges

$x=1$, get $\sum_{n=1}^{\infty} \frac{\overset{(-1)^n 3^n}{(-3)^n}}{3^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by alternating series test

Interval of convergence is $[1, 7)$

6. (10pts) Justify why the series converges and find its sum.

$$\sum_{n=1}^{\infty} \frac{3 \cdot 2^{2n}}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{3 \cdot (2^2)^n}{5 \cdot 5^n} = \sum_{n=1}^{\infty} \frac{3}{5} \cdot \left(\frac{4}{5}\right)^n = \frac{\text{First term}}{1-\frac{4}{5}} = \frac{\frac{3 \cdot 2^2}{5^2}}{\frac{1}{5}} = \frac{12}{5}$$

geometric series, $|r| = \left|\frac{4}{5}\right| = \frac{4}{5} < 1$, so converges

7. (24pts) The region bounded by the curves $y = 9 - x^2$ and $y = -3x + 9$ is rotated around the x-axis.

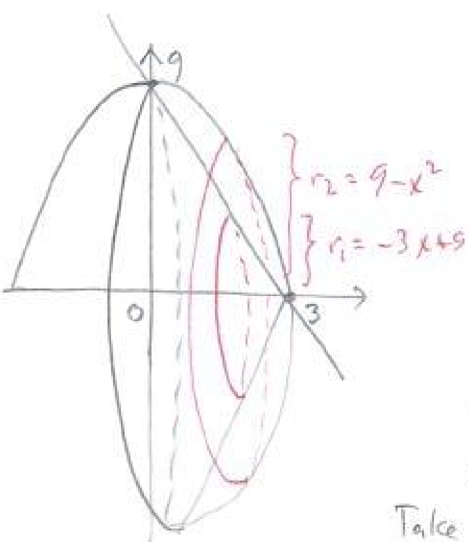
a) Sketch the solid and a typical cross-sectional washer.

b) Set up the integral for the volume of the solid.

c) On another picture, sketch the solid and a typical cylindrical shell.

d) Set up the integral for the volume of the solid using the shell method.

Simplify, but do not evaluate the integrals.

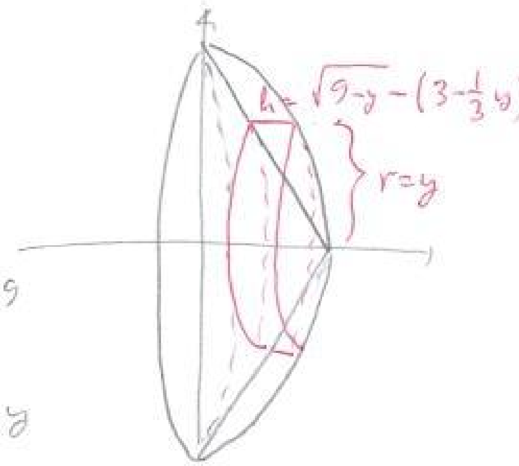


$9 - x^2 = -3x + 9$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0, 3$



$y = 9 - x^2$

$x^2 = 9 - y$

$x = \pm\sqrt{9-y}$

Take $x = \sqrt{9-y}$ since $x \geq 0$

$y = -3x + 9$

$3x = 9 - y$

$x = 3 - \frac{1}{3}y$

$$V = \int_0^3 \pi(r_2^2 - r_1^2)\, dx$$

$$= \int_0^3 \pi\left((9-x^2)^2 - (-3x+9)^2\right) dx$$

$$= \int_0^3 \pi\left(81 - 18x^2 + x^4 - (9x^2 - 54x + 81)\right) dx$$

$$= \pi \int_0^3 x^4 - 27x^2 + 54x\, dx$$

$$V = \int_0^9 2\pi r h\, dy$$

$$= \int_0^9 2\pi y\left(\sqrt{9-y} - \left(3 - \frac{1}{3}y\right)\right) dy$$

$$= 2\pi \int_0^9 y\sqrt{9-y} - 3y + \frac{1}{3}y^2\, dy$$

8. (16pts) Let $f(x) = \cos x$.

a) Find the 3rd Taylor polynomial for f centered at $a = \dfrac{\pi}{2}$.

b) Use Taylor's formula to get an estimate of the error $|R_3|$ on the interval $\left(\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right)$.

a)

n	$f^{(n)}$	$f^{(n)}(\frac{\pi}{2})$
0	$\cos x$	0
1	$-\sin x$	-1
2	$-\cos x$	0
3	$\sin x$	1
4	$\cos x$	

$$T_3(x) = \frac{-1}{1!}\left(x-\frac{\pi}{2}\right) + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3$$
$$= -\left(x-\frac{\pi}{2}\right) + \frac{1}{6}\left(x-\frac{\pi}{2}\right)^3$$

$$R_4(x) = \frac{\cos z}{4!}\left(x-\frac{\pi}{2}\right)^4$$

so $|R_4(x)| \le \dfrac{\frac{1}{2}}{4!}\left(\dfrac{\pi}{6}\right)^4 = \dfrac{\pi^4}{2\cdot 24\cdot 6^4} = \dfrac{\pi^4}{62208}$

On $[\frac{\pi}{3}, \frac{2\pi}{3}]$, $-\frac{1}{2} \le \cos z \le \frac{1}{2}$, $|x - \frac{\pi}{3}| \le \frac{\pi}{6}$

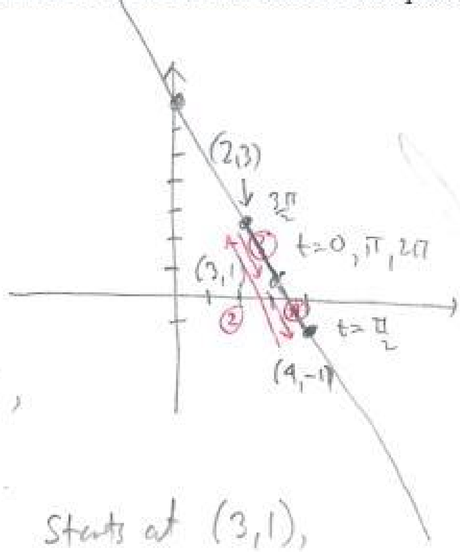
$36\cdot 36$
108
216
$1296 \cdot 48$
5184
10368 $+1 = 62,208$

9. (10pts) A particle moves along the path with parametric equations $x(t) = 3 + \sin t$, $y(t) = 1 - 2\sin t$, $0 \le t \le 2\pi$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

$x = 3 + \sin t \Rightarrow \sin t = x - 3$

$y = 1 - 2\sin t \Rightarrow y = 1 - 2(x-3)$

$y = -2x + 7$



For $0 \le t \le 2\pi$, $\sin t$ takes all values between -1 and 1,

so $3-1 \le 3+\sin t \le 3+1$

$2 \le 3 + \sin t \le 4$

Starts at $(3,1)$, moves along line $y = -2x+7$ to $(4,-1)$, then back to $(2,3)$, then returns to $(3,1)$

10. (24pts) The integral $\int_0^1 \cos\sqrt{x}\,dx$ is given. It cannot be found by antidifferentiation, since the antiderivative of $\cos\sqrt{x}$ is not expressible using elementary functions.

a) Write the expression you would use to calculate M_6, the midpoint rule with 6 subintervals. All the terms need to be explicitly written, do not use f in the sum.

b) It is known that $0.3 < y'' < \frac{1}{3}$ on $[0, 1]$: use it to find the error estimate for M_n in general.

c) What should n be in order for M_n to give you an error less than 10^{-4}?

d) Use a known power series for to find a power series for the above integral.

e) How many terms of the power series are needed to estimate the integral to accuracy 10^{-4}? Write the estimate as a sum (you do not have to simplify it).

f) Which method requires less computation to evaluate the integral with accuracy 10^{-4}, midpoint formula or series?

a) $\frac{1}{12}\quad \frac{1}{4}\quad \frac{5}{12}\quad \frac{7}{12}\quad \frac{3}{4}\quad \frac{11}{12}$ ← evaluation points.

$0\quad \frac{1}{6}\quad \frac{1}{3}\quad \frac{1}{2}\quad \frac{2}{3}\quad \frac{5}{6}\quad 1$

$$M_6 = \frac{1}{6}\left(f\left(\tfrac{1}{12}\right) + \dots + f\left(\tfrac{11}{12}\right)\right) = \frac{1}{6}\left(\cos\sqrt{\tfrac{1}{12}} + \cos\sqrt{\tfrac{1}{4}} + \cos\sqrt{\tfrac{5}{12}} + \cos\sqrt{\tfrac{7}{12}} + \cos\sqrt{\tfrac{3}{4}} + \cos\sqrt{\tfrac{11}{12}}\right)$$

b) $$|E| \le \frac{K_2(b-a)^3}{24n^2} = \frac{\frac{1}{3}(1-0)^3}{24n^2} = \frac{1}{72n^2}$$

c) Must have $\frac{1}{72n^2} \le 10^{-4}$

$$\frac{10^4}{72} \le n^2$$

$$n \ge \sqrt{\frac{10^4}{72}} = \frac{100}{6\sqrt{2}} \approx 12$$

$2.36 \qquad \approx 8.4$

d) $$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos\sqrt{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$$

$$\int_0^1 \cos\sqrt{x}\,dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)(2n)!}\Big|_0^1 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)(2n)!}$$

decreasing, goes to 0.

e) Need to have $\frac{1}{(n+1)(2n)!} < 10^{-4}$

n	$(n+1)(2n)!$
1	$2\cdot 2! = 4$
2	$3\cdot 4! = 72$
3	$4\cdot 6! = 4\cdot 720 < 10^4$
4	$5\cdot 8! = 5\cdot 40330 > 10^4$

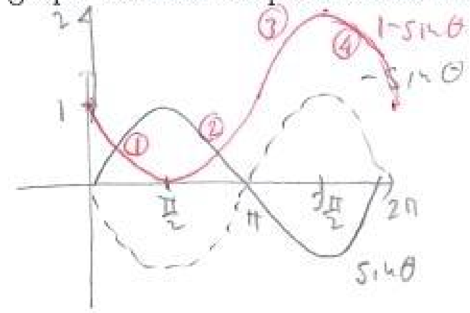
stop here →

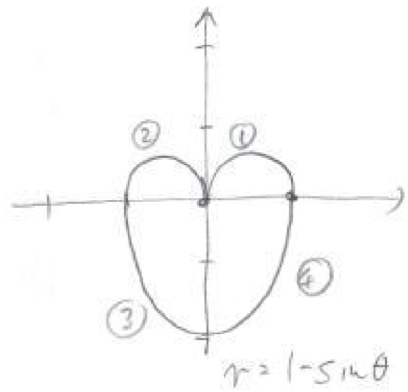
$720\cdot 7$
$5040\cdot 8$
40330

$$1 - \frac{1}{4} + \frac{1}{72} - \frac{1}{2880}$$ approx sum with accu. 10^{-4}

Series requires less computation, b/c midpoint formula requires computing cosine at 6 values, each involving a number of computations,

11. (12pts) First draw the graph of $r = 1 - \sin\theta$ in a cartesian θ-r coordinates. Use this graph to draw the polar curve with the same equation.





$$\frac{1-3(n-1)}{3} = \frac{-3n+4}{3}$$

Bonus (15pts) Find a fraction that is the approximation of $\sqrt[3]{9}$ with accuracy 10^{-3}. Start as below and take advantage of the binomial series.

$$\sqrt[3]{9} = \sqrt[3]{8\left(1+\frac{1}{8}\right)} = \sqrt[3]{8}\underbrace{\left(1+\frac{1}{8}\right)^{\frac{1}{3}}}_{\text{binomial series}} = 2\cdot\sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}\left(\frac{1}{8}\right)^n = 2\sum_{n=0}^{\infty}\frac{\frac{1}{3}\cdot\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\cdots\left(\frac{1}{3}-(n-1)\right)}{n!}\cdot\frac{1}{8^n}$$

$$= 2\sum_{n=0}^{\infty}(-1)^{n-1}\frac{1\cdot2\cdot5\cdots(3n-4)}{3^n\cdot8^n\cdot n!} =$$

$$= \sum_{n=0}^{\infty}(-1)^{n-1}\frac{2\cdot1\cdot2\cdot5\cdots(3n-4)}{24^n\cdot n!}$$ ← alternating series, decreasing

n	$\frac{2\cdot1\cdot2\cdot5\cdots(3n-4)}{24^n\cdot n!}$
1	$\frac{2}{24} = \frac{1}{12}$
2	$\frac{2\cdot2}{24^2\cdot2} = \frac{4}{24^2\cdot2} = \frac{1}{24\cdot8} = \frac{1}{192}$
3	$\frac{2\cdot1\cdot2\cdot5}{24^3\cdot6} = \frac{\cancel{2\cdot}2\cdot5}{24^2\cdot\cancel{4}6\cdot6} = \frac{5}{36\cdot24^2} = \frac{5}{20736} > 10^{-4}$
4	$\frac{2\cdot1\cdot2\cdot5\cdot8}{24^4\cdot24} = \frac{160}{24^5} < \frac{160}{20^5} = \frac{\cancel{160}}{\cancel{32}_{2}\cdot10^{\cancel{5}_{4}}} = \frac{1}{2\cdot10^4} < 10^{-4}$

$$2 - \frac{1}{12} + \frac{1}{192} - \frac{5}{20736}$$

$24^2 = 576$

$$\begin{array}{r} 576\cdot36 \\ \hline 1728 \\ 3456 \\ \hline 20736 \end{array}$$