Calculus 2 — Exam 5
MAT 308, Fall 2020 — D. Ivanšić

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Show all your work!

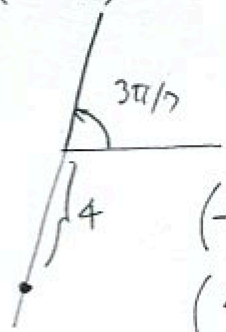
1. (12pts) Polar coordinates of two points are given.
a) Sketch the points in the plane.
b) For each point, give two additional polar coordinates, one with a negative r, one with a negative θ.

$\left(2, \frac{5\pi}{6}\right)$

$\left(-2, -\frac{\pi}{6}\right)$

$\left(2, -\frac{7\pi}{6}\right)$

$\left(-4, \frac{3\pi}{7}\right)$



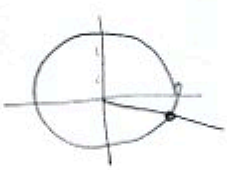
$\left(-4, \frac{17\pi}{7}\right)$

$\left(4, -\frac{4\pi}{7}\right)$

2. (10pts) Convert (a picture may help):
a) $\left(4, -\frac{\pi}{6}\right)$ from polar to rectangular coordinates
b) $(-2\sqrt{2}, -2\sqrt{2})$ from rectangular to polar coordinates

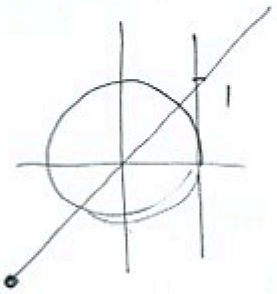
a) $x = 4\cos\left(-\frac{\pi}{6}\right) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$

$y = 4\sin\left(-\frac{\pi}{6}\right) = 4 \cdot \left(-\frac{1}{2}\right) = -2$



b) $\tan\theta = \frac{-2\sqrt{2}}{-2\sqrt{2}} = 1$

$r = \sqrt{(-2\sqrt{2})^2 + (-2\sqrt{2})^2} = \sqrt{16} = 4$



$\tan\theta = 1$, $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

choose this one, since point is in 3rd quadrant

$\left(4, \frac{3\pi}{4}\right)$

3. (14pts) Find the equation of the tangent line to the parametric curve $x = e^t \cos t$, $y = e^t \sin t$ at the point where $t = 0$.

$x' = e^t \cos t - e^t \sin t$

$y' = e^t \sin t + e^t \cos t$

$y - 0 = 1 \cdot (x - 1)$

$y = x - 1$

$x(0) = 1$

$y(0) = 0$

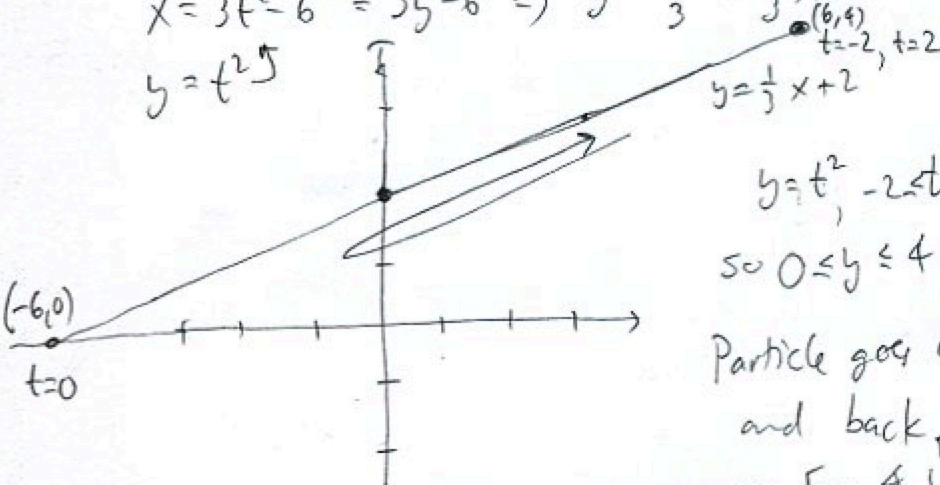
$x'(0) = 1 \qquad m = \frac{y'(0)}{x'(0)} = \frac{1}{1} = 1$

$y'(0) = 1$

4. (12pts) A particle moves along the path with parametric equations $x(t) = 3t^2 - 6$, $y(t) = t^2$ for $-2 \le t \le 2$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

$x = 3t^2 - 6 = 3y - 6 \Rightarrow y = \frac{x+6}{3} = \frac{1}{3}x + 2$

$y = t^2$

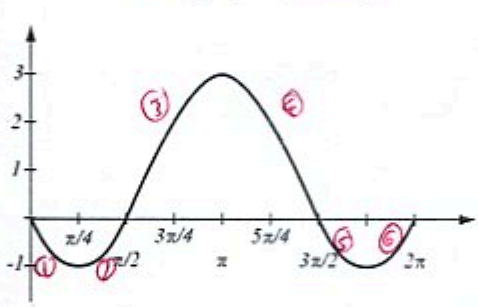


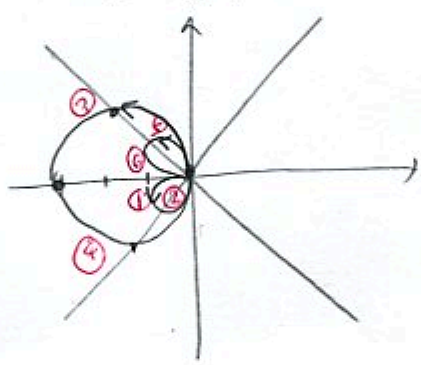
$y = t^2, \; -2 \le t \le 2$

so $0 \le y \le 4$

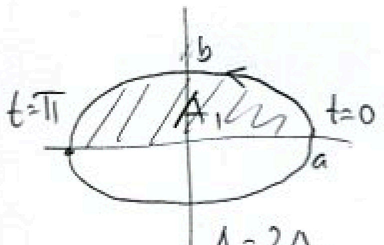
Particle goes once from (6, 4) to (-6, 0) and back, because y coordinates go from 4 to 0 to 4 as t goes from -2 to 2

5. (12pts) The graph of $r = f(\theta)$ is given in cartesian coordinates. Use its intervals of increase and decrease to help you sketch the polar curve $r = f(\theta)$. Indicate which piece of the cartesian graph corresponds to which piece of the polar graph.





6. (12pts) Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, given by parametric equations $x(t) = a\cos t$, $y(t) = b\sin t$, $0 \le t \le 2\pi$, where a and b are constants.



$A = 2A_1$

$$A_1 = \int_0^{\pi} y(t)\, x'(t)\, dt$$

$$= \int_0^{\pi} b\sin t\,(-a\sin t)\, dt = -ab\int_0^{\pi} \sin^2 t\, dt$$

$$= -ab\int_0^{\pi} \frac{1-\cos(2t)}{2}\, dt = -\frac{ab}{2}\Big(\pi - \underbrace{\frac{\sin(2t)}{2}\Big|_0^{\pi}}_{=0}\Big)$$

$$= -\frac{ab\pi}{2}$$

Since parametrization traces curve with decreasing x's,

area is $2 \cdot \frac{ab\pi}{2} = ab\pi$.

7. (10pts) In celebration of the end of the semester, an empty tub of disinfectant wipes is launched from the roof of Faculty Hall, so that its position is given by $x(t) = 10t$, $y(t) = 20 + 15t - 5t^2$, where length is measured in meters, time in seconds.

a) Splat! When does the tub hit the ground?

b) What horizontal distance did it travel until touchdown?

a) $y = 0$

$20 + 15t - 5t^2 = 0 \quad | \div (-5)$

$t^2 - 3t + 4 = 0$

$(t-4)(t+1) = 0$

$t = -1, 4$

After 4 seconds

b) $x(4) = 40$ meters

8. (18pts) A parametric curve is given by $x(t) = t^3 - 12t$, $y(t) = t^2 + 2t - 8$.
a) Find the points on the curve where the tangent line is horizontal or vertical.
b) Where does the curve go as $t \to \infty$ and $t \to -\infty$? (That is, find $\lim_{t\to\pm\infty} x(t)$, $\lim_{t\to\pm\infty} y(t)$.)
c) Plot the points from a) on a coordinate system and use them, along with information from b), or from plotting additional points, to get a graph of the curve. Recall that the curve moves in only one of general directions ↗ ↖ ↙ ↘ between points from a).

a) $x'(t) = 3t^2 - 12$

$y'(t) = 2t + 2$

$3t^2 - 12 = 0 \qquad 2t + 2 = 0$

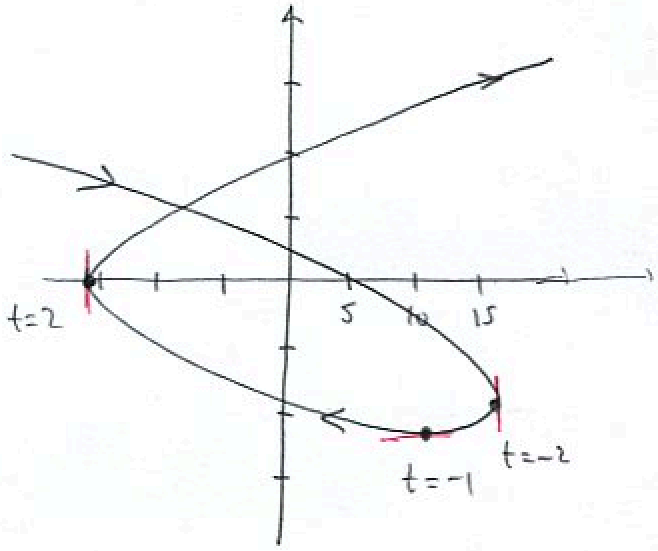
$t^2 = 4 \qquad t = -1$

$t = \pm 2$

b) $\lim_{t\to\pm\infty} t^3 - 12t = \lim_{t\to\pm\infty} t^3 \underbrace{\left(1 - \frac{12}{t^2}\right)}_{\to 1} = \pm\infty$

$\lim_{t\to\pm\infty} t^2 + 2t - 8 = \lim_{t\to\pm\infty} t^2 \underbrace{\left(1 - \frac{2}{t} - \frac{8}{t^2}\right)}_{\to 1} = \infty$

t	x	y	
-2	16	-8	vert
-1	11	-9	horiz.
2	-16	0	vert

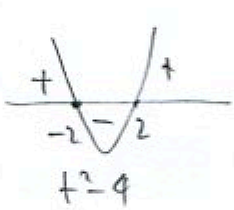


Bonus. (10pts) Find the intervals of concavity for the parametric curve given in problem 8.

$$F''(t) = \frac{\frac{d}{dt}\frac{y'}{x'}}{x'} = \frac{\frac{d}{dt}\frac{2t+2}{3t^2-12}}{3t^2-12} = \frac{\frac{2}{3}\frac{d}{dt}\frac{t+1}{t^2-4}}{3(t^2-4)} = \frac{\frac{2}{3}\cdot\frac{1\cdot(t^2-4)-(t+1)(2t)}{(t^2-4)^2}}{3(t^2-4)}$$

$$= \frac{2(t^2-4-2t^2-2t)}{9(t^2-4)^3} = \frac{2(-t^2-2t-4)}{9(t^2-4)^3} = -\frac{2(t^2+2t+4)}{9(t^2-4)^3} = -\frac{2\overbrace{((t+1)^2+3)}^{>0}}{9(t^2-4)^3}$$

Sign depends only on t^2-4 and is opposite of sign of t^2-4



	-2		2	
f''	$-$	$+$		$-$
f	CD	CU		CD