Calculus 2 — Exam 4
MAT 308, Fall 2020 — D. Ivanšić

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Show all your work!

Find the intervals of convergence for the series below. Don't forget to check the endpoints.

1. (16pts) $\sum_{n=0}^{\infty} 3^n \cdot \sqrt{n} \cdot (x-4)^n$

Root test: $\sqrt[n]{|3^n \sqrt{n}(x-4)^n|} = \sqrt[n]{3^n \sqrt{n}\,|x-4|^n} = 3 \cdot \underbrace{\sqrt[n]{\sqrt{n}}}_{\to 1} \cdot |x-4| \to 3 \cdot 1 \cdot |x-4|$

$3|x-4| < 1$

$|x-4| < \frac{1}{3}$

$-\frac{1}{3} \quad +\frac{1}{3}$

$\frac{11}{3} \quad 4 \quad \frac{13}{3}$

$\lim_{n\to\infty} \sqrt{n} = \infty$, $\lim_{n\to\infty} (-1)^n \sqrt{n}$ dne

When $x = \frac{13}{3}$, get $\sum_{n=0}^{\infty} 3^n \sqrt{n} \left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \sqrt{n}$

$x = \frac{11}{3}$, get $\sum_{n=0}^{\infty} 3^n \sqrt{n} \left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \sqrt{n}$

In both cases, $\lim a_n \neq 0$ so series diverge by divergence test

2. (10pts) $\sum_{n=1}^{\infty} \frac{e^n}{(2n)!} x^n$

Ratio test: $\left| \dfrac{\frac{e^{n+1}}{(2(n+1))!} x^{n+1}}{\frac{e^n}{(2n)!} x^n} \right| = \left| \frac{\cancel{e^{n+1}}^{\,1} \cancel{x^{n+1}}^{\,1}}{\underbrace{(2n+2)!}_{(2n+2)(2n+1)(2n)!}} \cdot \frac{(2n)!}{\cancel{e^n} \cancel{x^n}} \right| = \left| \frac{ex}{(2n+2)(2n+1)} \right|$

$= \frac{e|x|}{(2n+2)(2n+1)} \to \frac{e|x|}{\infty} = 0$ $\quad 0 < 1$ so series converges for every x.

3. (6pts) Use a known power series to find the sum:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n3^n} = \ln\left(1+\frac{1}{3}\right) = \ln\frac{4}{3}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad \text{series has } x = \frac{1}{3}$$

4. (8pts) Use a known power series to find the limit.

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x^3} = \lim_{x\to 0} \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}\right) - 2x}{x^3}$$

$$= \lim_{x\to 0} \frac{2x + \frac{2x^3}{6} + \frac{2x^5}{120} + \dots - 2x}{x^3} = \lim_{x\to 0} \frac{\frac{x^3}{3} + \frac{x^5}{60} + \dots}{x^3}$$

$$= \lim_{x\to 0} \left(\frac{1}{3} + \frac{x^2}{60} + \dots\right) = \frac{1}{3}$$

5. (14pts) Use geometric series and differentiation to get a power series for $\frac{x^2}{(1-x^2)^2}$. State the interval of convergence (no need to check the endpoints).

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n} \quad \leftarrow \text{converges when } |x^2| < 1$$

$$|x| < 1$$

$\frac{d}{dx}$

$$(-1)(1-x^2)^{-2} \cdot (-2x) = \sum_{n=0}^{\infty} 2n x^{2n-1}$$

$$(-1, 1)$$

$$\frac{2x}{(1-x^2)^2} = \sum_{n=1}^{\infty} 2n x^{2n-1} \quad \Big| \cdot \frac{x}{2}$$

$$\frac{x^2}{(1-x^2)^2} = \sum_{n=1}^{\infty} 2n x^{2n-1} \cdot \frac{x}{2} = \sum_{n=1}^{\infty} n \cdot x^{2n}$$

6. (12pts) Use known power series to show that $\frac{d}{dx}\cos x = -\sin x$.

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{2n\,x^{2n-1}}{(2n)(2n-1)!} = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!}$$

For $n=0$ this is 0

$$= \begin{bmatrix} n=k+1 \\ n=1,\ k=0 \\ 2n-1 = 2(k+1)-1 = 2k+1 \end{bmatrix} = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{x^{2k+1}}{(2k+1)!} = (-1)\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}.$$

$$= -\sin x$$

7. (18pts) Let $f(x) = \sqrt[3]{x}$.
a) Find the 2nd Taylor polynomial for f centered at $a = 8$.
b) Use Taylor's formula to get an estimate of the error $|R_2|$ on the interval $[6.5, 9.5]$. Leave your answer as a fraction.

a)

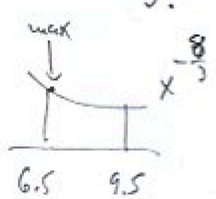
	eval at 8
$y = x^{\frac{1}{3}}$	$8^{\frac{1}{3}} = 2$
$y' = \frac{1}{3}x^{-\frac{2}{3}}$	$\frac{1}{3}8^{-\frac{2}{3}} = \frac{1}{3}2^{-2} = \frac{1}{12}$
$y'' = \frac{1}{3}\left(-\frac{2}{3}\right)x^{-\frac{5}{3}}$	$-\frac{2}{9}8^{-\frac{5}{3}} = -\frac{2}{9}2^{-5} = -\frac{2}{9\cdot 2^5} = -\frac{1}{9\cdot 2^4} = -\frac{1}{144}$
$y''' = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)x^{-\frac{8}{3}}$	$\frac{10}{27}x^{-\frac{8}{3}}$

$$T_2(x) = 2 + \frac{\frac{1}{12}}{1!}(x-8) - \frac{\frac{1}{144}}{2!}(x-8)^2 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

$$R_2(x) = \frac{\frac{10}{27}z^{-\frac{8}{3}}}{3!}(x-8)^3 \qquad |R_2(x)| \le \frac{\frac{\cancel{10}\,5}{27}\cdot 6.5^{-\frac{8}{3}}}{\cancel{6}\,3}\cdot\left(\frac{3}{2}\right)^3 = \frac{5}{\cancel{3^4}\,1}\cdot\frac{\cancel{3^3}}{2^3}\cdot\frac{1}{6.5^{8/3}} = \boxed{\frac{5}{24\cdot 6.5^{8/3}}}$$

$$|x-8| \le 1.5 = \frac{3}{2}$$

error estimate



8. (16pts) Use the known power series for $\cos x$ to find the series representing $\int_0^{\frac{1}{2}} \cos\sqrt{x}\,dx$. (Note that $\cos\sqrt{x}$ does not have an antiderivative that is an elementary function.) Give an approximation of the definite integral with accuracy 10^{-3}. Write the approximation as a sum (you do not have to simplify it).

$$\cos\sqrt{x} = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$$

$$\int_0^{\frac{1}{2}} \cos\sqrt{x}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)\cdot(2n)!}\Big|_0^{\frac{1}{2}} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)2^{n+1}\cdot(2n)!}$$

alternating series with decreasing terms, error estimate applies.

n	$(n+1)\cdot 2^{n+1}(2n)!$
1	$2\cdot 2^2\cdot 2! = 16$
2	$3\cdot 2^3\cdot 4! = 24\cdot 24 = 576$
3	$4\cdot 2^4\cdot 6! = 64\cdot 720 > 10^3$

$S_2 = \frac{1}{2} - \frac{1}{16} + \frac{1}{576}$ approx. integral with accu. 10^{-3}

Bonus (10pts) Find a fraction that is the approximation of $\sqrt{5}$ with accuracy 10^{-2}. Start as below and take advantage of the binomial series.

$$\sqrt{5} = \sqrt{4\left(1+\frac{1}{4}\right)} = 2\left(1+\frac{1}{4}\right)^{\frac{1}{2}} = 2\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\cdot\frac{1}{4^n} = \sum_{n=0}^{\infty} 2\cdot\frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-(n-1)\right)}{n!\cdot 4^n}$$

$$\frac{1-2(n-1)}{2} = -\frac{2n-3}{2}$$

$$= 2 + \frac{1}{1\cdot 4} + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!\,4^n\,2^{n-1}} = 2 + \frac{1}{4} + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!\cdot 2^{3n-1}}$$

alternating, terms decreasing.

n	$\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!\cdot 2^{3n-1}}$
2	$\frac{1}{2\cdot 2^5} = \frac{1}{64}$
3	$\frac{1\cdot 3}{6\cdot 2^8} = \frac{1}{2^9} = \frac{1}{512} < 10^{-2}$

$S = 2 + \frac{1}{4} - \frac{1}{64} = \frac{128+16-1}{64} = \frac{143}{64}$