Calculus 2 — Exam 3
MAT 308, Spring 2020 — D. Ivanšić

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Show all your work!

Find the limits, if they exist.

1. (6pts) $\lim_{n\to\infty} \frac{n}{1.5^n} = \lim_{x\to\infty} \frac{x}{1.5^x} \overset{LH}{=} \lim_{x\to\infty} \frac{1}{\ln 1.5 \cdot 1.5^x} = \frac{1}{\underbrace{\ln 1.5}_{>0} \cdot \infty} = \frac{1}{\infty} = 0$

2. (6pts) $\lim_{n\to\infty}(2+(-1)^n) =$ Does not exist

Sequence is $1, 3, 1, 3, 1, 3, \ldots$

does not approach any single real number.

3. (10pts) Find the limit. Use the theorem that rhymes with the insects typically found on cats and dogs.

$$\lim_{n\to\infty} \frac{2^n(\sin(17n)+1)}{3^n}$$

$-1 \le \sin(17n) \le 1$

$0 \le \sin(17n)+1 \le 2 \quad \Big| \left(\frac{2}{3}\right)^n$

$0 \le \left(\frac{2}{3}\right)^n(\sin(17n)+1) \le 2\cdot\left(\frac{2}{3}\right)^n$

$\lim_{n\to\infty} 0 = 0$

$\lim_{n\to\infty} 2\cdot\left(\frac{2}{3}\right)^n = 2\cdot 0 = 0$

Are equal, so by the squeeze theorem

$\lim_{n\to\infty}\left(\frac{2}{3}\right)^n(\sin(17n)+1) = 0$

4. (6pts) Write the series using summation notation:

$$\frac{3}{8} - \frac{7}{16} + \frac{11}{32} - \frac{15}{64} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{3+4n}{8 \cdot 2^n}$$

$n = 0 \quad 1 \quad 2 \quad 3$

5. (12pts) Justify why the series converges and find its sum.

geom. series

$$\sum_{n=2}^{\infty} \frac{2 \cdot 3^{2n-1}}{2^{4n+3}} = \sum_{n=2}^{\infty} \frac{2 \cdot 3^{2n} \cdot 3^{-1}}{2^{4n} \cdot 2^3} = \sum_{n=2}^{\infty} \frac{2 \cdot 3^{-1}}{2^3} \left(\frac{3^2}{2^4}\right)^n = \left[\frac{\text{first term}}{1-r} \right]$$

$r = \frac{9}{16} < 1$

$$= \frac{\frac{\cancel{2} \cdot 3^3}{2^{\cancel{11}\,10}}}{1 - \frac{9}{16}} = \frac{3^3}{2^{\cancel{10}\,6}} \cdot \frac{16}{7} = \frac{27 \cdot}{2^6 \cdot 7} = \frac{27}{448}$$

$\cancel{24}$ 64

Determine whether the following series converge and justify your answer.

6. (6pts) $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{n} = \sum_{n=1}^{\infty} \left(2 - \frac{1}{n}\right)$

$\lim_{n \to \infty} 2 - \frac{1}{n} = 2 \neq 0$ so diverges by divergence test

7. (12pts) $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sqrt[n]{5}}{3n^2}$

$\sqrt[n]{5} \leq 5$ for $n \geq 1$

$$\frac{1+\sqrt[n]{5}}{3n^2} \leq \frac{1+5}{3n^2} = \frac{6}{3n^2} = \frac{2}{n^2}$$

Since $\sum \frac{2}{n^2} = 2 \sum \frac{1}{n^2}$ is a convergent p-series $(p=2>1)$

given series converges by comparison test

8. (20pts) Consider the alternating series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n+\sqrt{n}}$.

a) Is the series convergent? Justify.
b) Is the series absolutely convergent? Justify.

a) Use alt. series test, $n+\sqrt{n}$ is increasing, so $\frac{1}{n+\sqrt{n}}$ is decreasing

$\left((x+\sqrt{x})' = 1 + \frac{1}{2\sqrt{x}} > 0 \text{ for } x \geq 0 \right)$

$\lim_{n\to\infty} \frac{1}{n+\sqrt{n}} = \frac{1}{\infty+\infty} = 0$

Series converges by alternating series test

b) $\sum \frac{1}{n+\sqrt{n}}$ is like $\sum \frac{1}{n}$ by dominant terms

$$\frac{\frac{1}{n+\sqrt{n}}}{\frac{1}{n}} = \frac{n}{n+\sqrt{n}} = \frac{\cancel{n}^{\,1}}{\cancel{n}(1+\underbrace{\tfrac{1}{\sqrt{n}}}_{\to 0})} \to \frac{1}{1+0} = 1 \neq 0, \infty$$

Since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n+\sqrt{n}}$ diverges by limit comparison test.

Determine whether the following series converge using the root or ratio test.

9. (11pts) $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{22^n}{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}$

Ratio test:

$$\left| \frac{(-1)^n \frac{22^{n+1}}{1\cdot3\cdot5\cdot\ldots(2(n+1)-1)}}{(-1)^{n-1}\frac{22^n}{1\cdot3\cdot5\cdot\ldots\cdot 2n-1}} \right| = \frac{22^{\cancel{n+1}\,1}}{\cancel{1\cdot3\cdot5\cdots(2n-1)}(2n+1)} \cdot \frac{\cancel{1\cdot3\cdot5\cdots(2n-1)}}{\cancel{22^n}}$$

$$= \frac{22}{2n+1} \to \frac{22}{\infty} = 0$$

$0<1$ so series converges absolutely by ratio test.

10. (11pts) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4+3n^2+1}{2^{3n+1}}$

$\sqrt[n]{\text{polynomial}(n)} \to 1$

Root test: $\sqrt[n]{\left|\underbrace{\frac{n^4+3n^2+1}{2^{3n+1}}}_{>0}\right|} = \frac{\sqrt[n]{n^4+3n^2+1}}{\sqrt[n]{2\cdot 2^{3n}}} = \frac{\sqrt[n]{n^4+3n^2+1}}{\sqrt[n]{2}\cdot 2^3} \to \frac{1}{1\cdot 8} = \frac{1}{8}$

$(2^{3n})^{\frac{1}{n}} = 2^3 \qquad \sqrt[n]{c} \to 1$

Since $\frac{1}{8} < 1$, series converges absolutely by root test.

Bonus. (10pts) Play this game on a basic calculator: enter any positive number, then keep pressing the $\sqrt{\ }$ key. After a while, the display stabilizes at a number. (In case you have never used a basic calculator, pressing $\sqrt{\ }$ immediately returns the square root of the number.)

a) Use a sequence and a limit to explain what is happening.

b) At which number does the display stabilize?

a) Let $a_0 > 0$ be the initial number, a_n = the number after $\sqrt{\ }$ is pressed n times.

Then $a_1 = \sqrt{a_0}$, $a_2 = \sqrt{a_1} = \sqrt{\sqrt{a_0}} = a_0^{\frac{1}{4}}$, $a_3 = \sqrt{a_2} = (a_0^{\frac{1}{4}})^{\frac{1}{2}} = a_0^{\frac{1}{8}}$, etc.

We see that $a_n = a_0^{\frac{1}{2^n}}$, so $\lim_{n\to\infty} a_0^{\frac{1}{2^n}} = a_0^{\frac{1}{\infty}} = a_0^{0} = 1$

b) It stabilizes at the limit, which is 1.