Calculus 2 — Exam 2	
MAT 308, Spring 2020 — D. Ivanšić	

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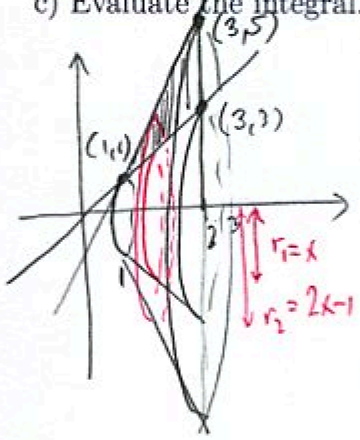
1. (24pts) The region bounded by the lines $y = x$ and $y = 2x - 1$ and $x = 3$ is rotated around the x-axis.

a) Sketch the solid and a typical cross-sectional washer.

b) Set up the integral for the volume of the solid.

c) Evaluate the integral.



$2x - 1 = x$

$x = 1$

$$V = \int_1^3 \pi(2x-1)^2 - \pi x^2\,dx$$

$$= \int_1^3 \pi(4x^2 - 4x + 1 - x^2)\,dx$$

$$= \pi\int_1^3 3x^2 - 4x + 1\,dx$$

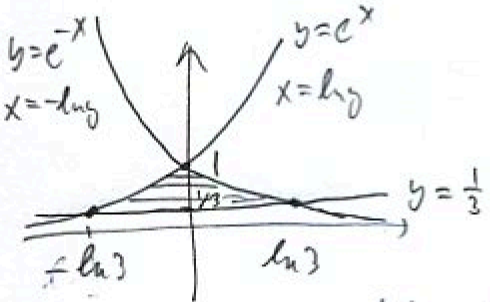
$$= \pi(x^3 - 2x^2 + x)\Big|_1^3 = \pi(27 - 1 - 2(9-1) + 3 - 1)$$

$$= \pi(26 - 16 + 2) = 12\pi$$

2. (14pts) Consider the region bounded by curves $y = e^x$, $y = e^{-x}$ and $y = \frac{1}{3}$.

a) Sketch the region.

b) Set up the integral that computes its area. Simplify, but do not evaluate the integral.



Integrate by y:

$$\int_{\frac{1}{3}}^{1} -\ln y - \ln y\,dy = \int_{\frac{1}{3}}^{1} -2\ln y\,dy$$

$e^x = \frac{1}{3}$

$x = \ln\frac{1}{3} = -\ln 3$

$e^{-x} = \frac{1}{3}$

$-x = \ln\frac{1}{3}$

$x = \ln 3$

$y = e^{\pm x}$

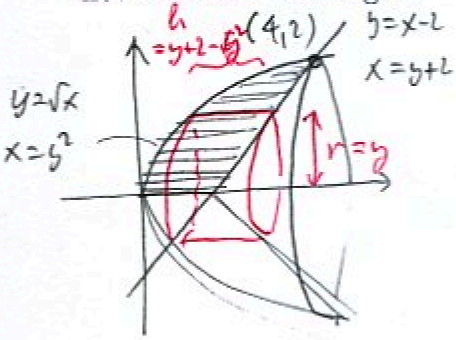
$\pm x = \ln y$

$x = \pm\ln y$

Integrating by x requires two integrals:

$$\int_{-\ln 3}^{0} e^x - \frac{1}{3}\,dx + \int_0^{\ln 3} e^{-x} - \frac{1}{3}\,dx$$

3. (16pts) Rotate the region bounded by the curves $y = \sqrt{x}$, $y = x - 2$ and the x-axis about the x-axis to get a solid.
a) Sketch the solid and a typical cylindrical shell.
b) Set up the integral for the volume of the solid using the shell method. Simplify, but do not evaluate the integral.



$$V = \int_a^b S(y)\,dy = \int_0^2 S(y)\,dy = \int_0^2 2\pi r h\,dy$$

$$= \int_0^2 2\pi y\,(y+2-y^2)\,dy$$

$$= 2\pi \int_0^2 -y^3+y^2+2y\,dy$$

$\sqrt{x} = x-2 \quad |^2$

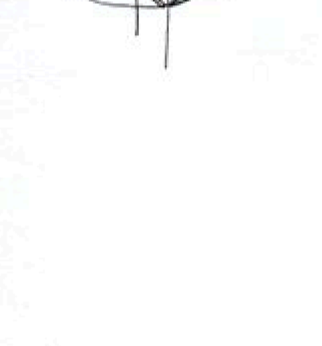
$x = x^2-4x+4$

$x^2-5x+4=0$

$(x-1)(x-4)=0$

$x = 1, 4$ doesn't satisfy original eq

4. (18pts) Consider the arc of the circle $x^2 + y^2 = 9$ that is to the right of the line $x = 1$. Rotate it around around the x-axis to get a spherical cap. Find the surface area of this cap.



$$S = \int_a^b 2\pi r\,ds = \int_1^3 2\pi\sqrt{9-x^2}\sqrt{1+\left(\sqrt{9-x^2}\,'\right)^2}\,dx$$

$$= 2\pi\int_1^3 \sqrt{9-x^2}\sqrt{1+\left(\frac{-2x}{2\sqrt{9-x^2}}\right)^2}\,dx = 2\pi\int_1^3 \sqrt{9-x^2}\sqrt{1+\frac{x^2}{9-x^2}}$$

$$= 2\pi\int_1^3 \sqrt{9-x^2}\sqrt{\frac{9-x^2+x^2}{9-x^2}}\,dx = 2\pi\int_1^3 \sqrt{9}\,dx = 2\pi\cdot 3\cdot(3-1)$$

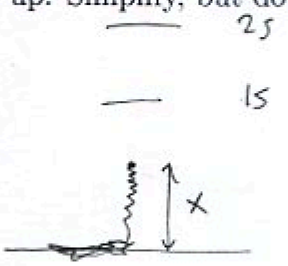
$$= 12\pi$$

5. (12pts) Set up the integral for the length of the curve $y = x^3 - 4x$ from $x = 0$ to $x = 2$. Simplify, but do not evaluate the integral.

$$L = \int_0^2 \sqrt{1 + f'(x)^2}\, dx = \int_0^2 \sqrt{1 + (3x^2 - 4)^2}\, dx = \int_0^2 \sqrt{1 + 9x^4 - 24x^2 + 16}\, dx$$

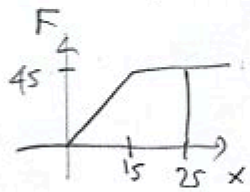
$$= \int_0^2 \sqrt{9x^4 - 24x^2 + 17}\, dx$$

6. (16pts) A 15-foot chain weighing 45lbs is lying on the ground. Set up the computation for the work done if one of its ends is hoisted to height 25 feet, pulling the rest of the chain up. Simplify, but do not evaluate any integrals.



Force needed to lift chain when top is x ft high is

$$\text{is } \begin{cases} 3x & \text{if } 0 \le x \le 15 \\ 45 & \text{if } 15 \le x \le 25 \end{cases}$$



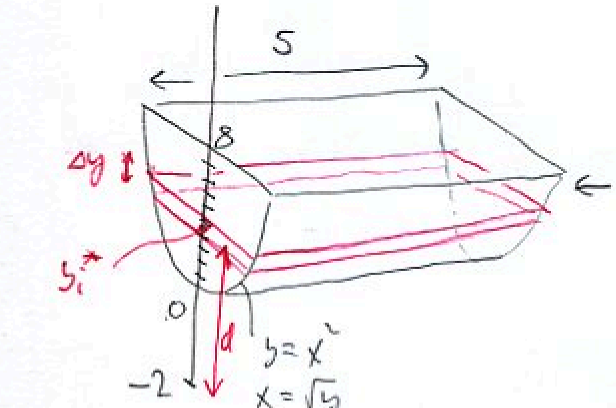
$$\frac{45\,lb}{15\,ft} = 3\,lb/ft$$

$$W = \int_0^{25} F(x)\, dx = \int_0^{15} 3x\, dx + \int_{15}^{25} 45\, dx$$

$45 \cdot (25 - 15)$

$$= \int_0^{15} 3x\, dx + 450$$

Bonus (10pts) Cross sections by parallel vertical planes of a 5-meter long tank are like the region between the curves $y = x^2$ and $y = 8$. (Thus, the tank is 8 meters tall and has a flat top.) Set up the integral for the work needed to fill this tank with water, assuming the water is pumped from 2 meters below the tank. Assume $g = 10$ and water density $= 1000\text{kg/m}^3$. Do not evaluate the integral, but do draw copious pictures!



d= distance each slice is lifted

work needed to fill this slice, pumping from level $y=-2$

$$F = \underbrace{\text{volume}}\cdot\text{density}\cdot g$$

$$= \overbrace{5\cdot 2\sqrt{y_i^*}\cdot\Delta y}\cdot 1000\cdot 10$$

$$d = y-(-2) = y+2$$

$$W \approx \sum_{i=1}^{n} 100000\sqrt{y_i^*}\,(y+2)\,\Delta y$$

$$\Rightarrow \quad W = \int_0^8 100000\sqrt{y}\,(y+2)\,dy$$