Calculus 2 — Exam 1
MAT 308, Spring 2020 — D. Ivanšić

Name:

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Find the following integrals:

1. (7pts) $\int xe^{3x}\,dx = \begin{bmatrix} u=x & dv=e^{3x}dx \\ du=dx & v=\frac{e^{3x}}{3} \end{bmatrix} = \frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3}\,dx$

$= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C$

2. (7pts) $\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}\,dx = \frac{1}{2}\int 1-\cos(2x)\,dx$

$= \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right)$

$= \frac{x}{2} - \frac{\sin(2x)}{4} + C$

Determine whether the following improper integral converges, and, if so, evaluate it. (Calculate directly, comparison would be hard.)

3. (14pts) $\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2}\,dx = 1$

$\int_1^t \frac{\ln x}{x^2}\,dx = \begin{bmatrix} u=\ln x & dv=\frac{1}{x^2}dx \\ du=\frac{1}{x}dx & v=-\frac{1}{x} \end{bmatrix} = -\frac{\ln x}{x}\Big|_1^t - \int_1^t -\frac{1}{x^2}\,dx$

$= -\left(\frac{\ln t}{t} - \frac{\ln 1}{1}\right) + \left(-\frac{1}{x}\right)\Big|_1^t = -\frac{\ln t}{t} - \frac{1}{t} + \frac{1}{1}$

$\lim_{t\to\infty} 1 - \frac{\ln t}{t} - \frac{1}{t} = 1 - \lim_{t\to\infty} \frac{\frac{1}{t}}{1} = 1 - 0 = 1$

Use trigonometric substitution to evaluate the following integrals. Don't forget to return to the original variable where appropriate.

4. (14pts) $\displaystyle\int \frac{x^3}{\sqrt{x^2-1}}\,dx = \begin{bmatrix} x = \sec\theta \\ dx = \sec\theta\tan\theta \end{bmatrix} = \int \frac{\sec^3\theta}{\sqrt{\underbrace{\sec^2\theta-1}_{\tan^2\theta}}}\cdot\sec\theta\tan\theta\,d\theta$

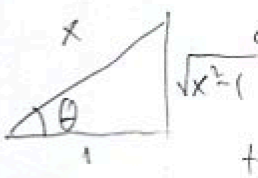
$$= \int \frac{\sec^4\theta\,\cancel{\tan\theta}}{\cancel{\tan\theta}} = \int \sec^4\theta\,d\theta = \int \underbrace{\sec^2\theta}_{\tan^2\theta+1}\sec^2\theta\,d\theta = \begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\,d\theta \end{bmatrix}$$

$$= \int (u^2+1)\,du = \frac{u^3}{3} + u = \frac{\tan^3\theta}{3} + \tan\theta = \frac{(x^2-1)^{\frac{3}{2}}}{3} + \sqrt{x^2-1} + C$$

$\sqrt{x^2-1}^{\,3}$

$\sec\theta = x$

$\cos\theta = \frac{1}{x}$



$\tan\theta = \sqrt{x^2-1}$

5. (14pts) $\displaystyle\int_0^{\frac{3}{2}} \frac{1}{(9-x^2)^{\frac{3}{2}}}\,dx = \begin{bmatrix} x = 3\sin\theta & x = \frac{3}{2},\ \frac{3}{2} = 3\sin\theta,\ \sin\theta = \frac{1}{2},\ \theta = \frac{\pi}{6} \\ dx = 3\cos\theta\,d\theta & x = 0,\ 0 = 3\sin\theta,\ \sin\theta = 0,\ \theta = 0 \end{bmatrix}$

$$= \int_0^{\pi/6} \frac{3\cos\theta}{(9-9\sin^2\theta)^{3/2}}\,d\theta = \int_0^{\pi/6} \frac{3\cos\theta}{(9\cos^2\theta)^{3/2}}\,d\theta = \int_0^{\pi/6} \frac{3\cos\theta}{27\cos^3\theta}\,d\theta = \frac{1}{9}\int_0^{\pi/6} \frac{1}{\cos^2\theta}\,d\theta$$

$= \sec^2\theta$

$9(1-\sin^2\theta)$
$= \cos^2\theta$

$$= \frac{1}{9}\tan\theta\Big|_0^{\pi/6} = \frac{1}{9}\left(\frac{1}{\sqrt{3}} - 0\right) = \frac{1}{9\sqrt{3}}$$

Use the method of partial fractions to find the following integrals.

6. (14pts) $\displaystyle\int \frac{-x^2-3x+2}{(x+1)(x^2+1)}\,dx = \int \frac{2}{x+1} + \frac{-3x}{x^2+1}\,dx = 2\ln|x+1| - \frac{3}{2}\ln|x^2+1| + C$

$$\frac{-x^2-3x+2}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{(x^2+1)} \quad \Big|\cdot (x+1)(x^2+1)^2$$

$$-x^2-3x+2 = A(x^2+1) + (Bx+C)(x+1)$$

x^0 $\quad 2 = A+C \xrightarrow{\text{subtract}} \quad C-B=3$

x^1 $\quad -3 = C+B \qquad\qquad C+B=-3$

x^2 $\quad -1 = A+B \qquad\qquad 2C=0$

$C=0 \Rightarrow B=-3 \Rightarrow A=2$

7. (10pts) Use comparison to determine whether the improper integral $\displaystyle\int_1^\infty \frac{x^2}{x^4+7}\,dx$ converges.

$\dfrac{x^2}{x^4+7} \approx \dfrac{x^2}{x^4} = \dfrac{1}{x^2}$ try for convergence

$\dfrac{x^2}{x^4+7} \leq \dfrac{x^2}{x^4} = \dfrac{1}{x^2}$

smaller denom

$\dfrac{x^2}{x^4+7} \leq \dfrac{1}{x^2}$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, so does $\int_1^\infty \frac{x^2}{x^4+7}$ by comparison theorem

8. (20pts) Suppose we wanted to approximate the number $\ln 5$. We could do it by approximating the integral $\int_1^4 \frac{1}{x}\,dx = \ln 4$, which uses only the four basic algebraic operations.
a) Write the expression you would use to calculate T_6, the trapezoid rule with 6 subintervals. All the terms need to be explicitly written, do not use f in the sum.
b) Find the error estimate for T_n in general. You will need the second derivative of $\frac{1}{x}$.
c) Estimate the error for T_6
d) What should n be in order for T_n to give you an error less than 10^{-4}?

a) $1 \;\; \frac{3}{2} \;\; 2 \;\; \frac{5}{2} \;\; 3 \;\; \frac{7}{2} \;\; 4$

$\Delta x = \frac{1}{2}$

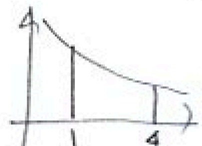
$$T_6 = \frac{1}{2}\cdot\frac{1}{2}\left(\frac{1}{1} + 2\cdot\frac{1}{\frac{3}{2}} + 2\cdot\frac{1}{2} + 2\cdot\frac{1}{\frac{5}{2}} + 2\cdot\frac{1}{3} + 2\cdot\frac{1}{\frac{7}{2}} + \frac{1}{4}\right)$$

in formula Δx

b) $y = \frac{1}{x} = x^{-1}$

$y' = -x^{-2}$

$y'' = 2x^{-3} = \frac{2}{x^3}$



decreasing, has max at $x = 1$

$y''(1) = 2$, so $K = 2$

$$|E_T| \le \frac{K\cdot(b-a)^3}{12n^2} = \frac{2\cdot(4-1)^3}{12n^2} = \frac{\cancel{2}\cdot\cancel{27}^{\,9}}{\cancel{12}_{\,2}n^2} = \frac{9}{2n^2}$$

c) For T_6, $n = 6$, $|E_T| \le \frac{9}{2\cdot 6^2} = \frac{9}{72} = \frac{1}{8}$

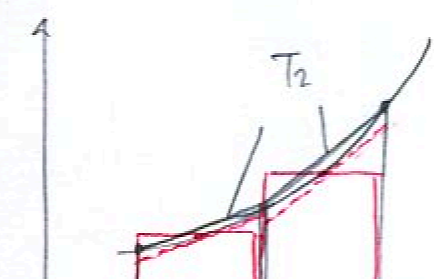
d) Must have

$$\frac{9}{2n^2} \le 10^{-4} \quad /\cdot 10^4 n^2$$

$$\frac{9\cdot 10^4}{2} \le n^2$$

$$n \ge \frac{3\cdot 10^2}{\sqrt{2}} = \frac{300}{\sqrt{2}}$$

Bonus (10pts) On the interval $[1, 3]$, draw a nice big picture of any concave upward function f whose graph is above the x-axis. Then draw the straight-edge shapes whose area is represented by the trapezoid and midpoint approximations T_2 and M_2 for the integral $I = \int_1^3 f(x)\,dx$. Put the numbers I, T_2 and M_2 in increasing order and justify this order precisely with your picture.



$M_2 < I < T_2$

Area of trapezoids associated with T_2 is clearly greater than area under curve, because the trapezoids cover it. Area of rectangles associated with M_2 is smaller than area under curve, because the red rectangles have the same area as the red trapezoids, which lie under the curve, so their area is less than area under curve.