Advanced Calculus 1 — Exam 1	**Name:** ____________________
MAT 525/625, Fall 2019 — D. Ivanšić	*Show all your work!*

Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define a denumerable set.

Theory 2. (3pts) State the multiplicative properties of $\mathbf{R}$. That is, state the four algebraic properties of $\mathbf{R}$ that deal only with multiplication.

Theory 3. (3pts) State the Completeness Property of $\mathbf{R}$.

Type A problems (5pts each)

A1. Show using Mathematical Induction that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. (Here i is the imaginary unit, $i^2 = -1$, and recall that $\sin(x+y) = \sin x\cos y + \cos x\sin y$ and $\cos(x+y) = \cos x\cos y - \sin x\sin y$).

A2. Let A and B be sets, where A is finite. Show: $A \cup B$ is finite if and only if B is finite. You may use the statement that the union of two disjoint finite sets is finite.

A3. If $a, b \in \mathbf{R}$, use only the axiomatic algebraic properties of $\mathbf{R}$ to show that $-(a+b) = (-a) + (-b)$, and $a \cdot (-b) = -ab$.

A4. Show that for all $x, y, z \in \mathbf{R}$, $||x-y| - |z-y|| \leq |x-z|$.

A5. If they exist, find a lower bound of S, an upper bound of S, $\inf S$ and $\sup S$. There is no need to justify. a) $S = \{x \mid x < 0\}$ b) $S = \left\{\frac{n-1}{n} \mid n \in \mathbf{N}\right\}$

Type B problems (8pts each)

B1. Show that the collection of two-element subsets of $\mathbf{N}$ is denumerable.

B2. Recall that we set $\frac{a}{b} = a \cdot \frac{1}{b}$. Use only the axiomatic algebraic properties of $\mathbf{R}$ to show the rule for adding fractions: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. (You will need to prove an intermediate lemma.)

B3. Show directly that $\sqrt{8}$ is not a rational number, that is, without resorting to knowledge that $\sqrt{2}$ is not rational.

B4. Determine and sketch the set of points in the plane satisfying $|x| - 2|y| \leq 4$.

B5. Consider the subset S of $\mathbf{R}$, $S = \{x \in \mathbf{Q} \mid x > \pi\}$. If they exist, find a lower bound of S, an upper bound of S, $\inf S$ and $\sup S$. Prove the details, including nonexistence of any of the quantities.

B6. Let $S \subset (0, \infty)$ be a nonempty set that is bounded above. Show that $\inf \frac{1}{S} = \frac{1}{\sup S}$. (As expected, $\frac{1}{S} = \{\frac{1}{s} \mid s \in S\}$.)

Type C problems (12pts each)

C1. Show that the set of all functions $\mathbf{N} \to \{0,1\}$ is uncountable. *Hint: any function* $a : \mathbf{N} \to \{0,1\}$ *may be thought of as a sequence of 0's and 1's.*

C2. Let S be a bounded nonempty subset of $\mathbf{R}$.
a) Show by example that, in general, $\sup|S| \neq |\sup S|$.
b) Figure out the correct expression for $\sup|S|$ and prove it. Some homework statements may be helpful here, and you may use them without proof.

Advanced Calculus 1 — Exam 2 **MAT 525/625, Fall 2019 — D. Ivanšić**

Name: ______________________________

Show all your work!

Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define when the limit of a sequence (x_n) is x.

Theory 2. (3pts) State the squeeze theorem for sequences.

Theory 3. (3pts) State the Cauchy Convergence Theorem.

Type A problems (5pts each)

A1. Use the definition of the limit to show $\lim \frac{n+3}{5n+4} = \frac{1}{5}$.

A2. Find $\lim \left(1 + \frac{1}{n+1}\right)^{n-1}$.

A3. Find $\lim \sqrt[n]{n^2 + 4n + 7}$.

A4. Use the "ratio test" or other method to find the limit of the sequence $\frac{b^n}{n!}$, $b > 0$.

A5. Prove the extended limit law $L + \infty = \infty$. That is, use the definition to show: if $\lim x_n = L \in \mathbf{R}$ and $\lim y_n = \infty$, then $\lim(x_n + y_n) = \infty$.

A6. Use the definition to show that the sequence $x_n = \frac{n}{n+6}$ is Cauchy.

Type B problems (8pts each)

B1. Let the sequence x_n be recursively given by: $x_1 = 1$, $x_{n+1} = 3 + \sqrt{2x_n + 1}$. Show that this sequence converges and find its limit.

B2. Prove the limit law $\lim(x_n \cdot y_n) = \lim x_n \cdot \lim y_n$.

B3. Let $a > 0$ and $b > 1$. Find $\lim \frac{n^a}{b^n}$.

B4. Show that if a sequence of positive numbers (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim x_{n_k} = \infty$.

B5. Let the sequence x_n be recursively given by: $x_1 > 0$, $x_{n+1} = \frac{1}{3+x_n}$. Show that this sequence is contractive and find its limit.

B6. Suppose that for two positive sequences (x_n) and (y_n), $\lim x_n y_n = \infty$ and (x_n) is bounded. Show that $\lim y_n = \infty$.

B7. Let $\lim x_n = x$ and $\lim y_n = y$, where $x < y$, $x, y \in \mathbf{R}$, and let $z_n = \max\{x_n, y_n\}$. Show that $\lim z_n = y$.

Type C problems (12pts each)

C1. Let (x_n) be a bounded sequence that does not converge to x. (It does not mean it converges to some other number.) Show (x_n) has a subsequence that converges to a number y, where $y \neq x$.

C2. Let $x_n = \sqrt[n]{n!}$.
a) Show (x_n) is increasing.
b) Show $n! \geq k!k^{n-k}$ for every $k = 1, 2, \ldots, n$.
c) Fixing a k, use b) to get an inequality that will give you the limit of (x_n).

Advanced Calculus 1 — Exam 3 MAT 525/625, Fall 2019 — D. Ivanšić

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Show all your work!

Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define when a function $f : A \to \mathbf{R}$ is bounded on a neighborhood of c, a cluster point of A.

Theory 2. (3pts) Define what $\lim_{x\to\infty} f(x) = L$ means.

Theory 3. (3pts) State the limit law for quotients.

Type A problems (8pts each)

A1. Find the limits, if they exist (just a computation is expected): a) $\lim_{x\to -2} \dfrac{x^2-3x-10}{x+2}$ b) $\lim_{x\to\infty} (\sqrt{x+7}-\sqrt{x+3})$.

A2. Does $\lim_{x\to\infty} \dfrac{\sin x + \cos x}{x}$ exist? If yes, find it, if not, justify.

A3. Show that $\lim_{x\to -1} \cos \dfrac{1}{x+1}$ does not exist.

A4. Use the definition to show that $\lim_{x\to 3} (2x+1) = 7$.

A5. Let $f(x) \geq 0$ and $L = \lim_{x\to c} f(x)$. Use the definition to show that $L \geq 0$, by showing that $L \geq -\epsilon$ for every $\epsilon > 0$.

Type B problems (8pts each)

B1. Use the definition to show that $\lim_{x\to c} \dfrac{1}{x^2+4} = \dfrac{1}{c^2+4}$.

B2. Use the definition to prove the extended limit law $\infty+\infty=\infty$, that is, if $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$, show that $\lim_{x\to c} (f(x)+g(x)) = \infty$.

B3. Suppose $f(x), g(x) > 0$, $\lim_{x\to c} f(x)g(x) = L$ and $\lim_{x\to c} f(x) = 0$. Show that $\lim_{x\to c} g(x) = \infty$.

B4. Find all the cluster points (in $\mathbf{R}$) of the set $(-\sqrt{2}, \sqrt{2}) \cap \mathbf{Q}$. You do not need to write a detailed proof, but justify your answer with a picture and a few words. Justify also why certain points are *not* cluster points of the set.

B5. Use the definition and properties of the sine function to show that $\lim_{x\to c} \sin x = \sin c$. You will find the formula $\sin u - \sin v = 2 \sin \frac{u-v}{2} \cos \frac{u+v}{2}$ useful.

B6. Suppose $f, g : (a,\infty) \to \mathbf{R}$, and $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to L} g(x) = \infty$. Show that $\lim_{x\to\infty} f(g(x)) = L$.

B7. Use a picture of the unit circle and areas to show that $\lim_{\theta \to 0} \cos\theta = 1$ by showing that $\lim_{\theta \to 0}(1 - \cos\theta) = 0$.

Type C problems (12pts each)

C1. Let $f : (0, \infty) \to \mathbf{R}$, $f(x) = 0$ if x is irrational, $f(x) = \frac{1}{n}$ if x is a rational number represented by $\frac{m}{n}$ in reduced form. Show that for every c, $\lim_{x \to c} f(x) = 0$.
Hint: show that in an open interval of width $\frac{1}{n}$ there are only finitely many rational numbers x so that $f(x) > \frac{1}{n}$ and take it from there.