Advanced Calculus 1 — Final Exam
MAT 525/625, Fall 2014 — D. Ivanšić

Name: ____________________

Show all your work!

Do all the theory problems. Then do at least five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) Define the supremum of a set.

Theory 2. (3pts) Define what $\lim_{x\to\infty} f(x) = L$ means.

Theory 3. (3pts) State the Bolzano-Weierstrass theorem for sequences.

Type A problems (5pts each)

A1. If they exist, find a lower bound of S, an upper bound of S, $\inf S$ and $\sup S$. There is no need to justify. a) $S = \mathbf{Q} \cap \left[\sqrt{5}, 7\right)$ b) $S = \left\{ \frac{(-1)^n}{n} + \frac{(-1)^m}{m} \mid n, m \in \mathbf{N} \right\}$

A2. Use the definition of the limit to show $\lim \frac{2n-4}{5n+2} = \frac{2}{5}$.

A3. Establish convergence or divergence of $x_n = \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{2n}}$.

A4. Find the limits, if they exist (just a computation is expected): a) $\displaystyle\lim_{x\to\infty} \frac{x-4}{\sqrt{x}-7}$ b) $\displaystyle\lim_{x\to 0+} \frac{\sqrt{5x+1}-1}{x^2}$.

A5. Let $f : \mathbf{R} \to \mathbf{R}$ be the function given by: $f(x) = x$, if x is rational, $f(x) = x^2$ if x is irrational. Find $\lim_{x\to 0} f(x)$ if it exists.

A6. Prove the extended limit law $\infty \cdot \infty = \infty$, where $L > 0$, that is, if $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$, show that $\lim_{x\to c}(f(x)g(x)) = \infty$.

Type B problems (8pts each)

B1. Let A be countable and B uncountable. Prove that $A \setminus B$ is countable and $B \setminus A$ is uncountable.

B2. Let S be a nonempty set that is bounded above. Show that $\sup S^3 = (\sup S)^3$. (As expected, $S^3 = \{s^3 \mid s \in S\}$.)

B3. Let the sequence x_n be recursively given by: $x_1 = 2$, $x_{n+1} = 5 - \frac{2}{x_n}$. Show that this sequence converges and find its limit.

B4. Use the definition to show that the sequence $x_n = \frac{n^2}{n^2+1}$ is Cauchy.

B5. Find $\lim \left(1 + \frac{1}{n^2}\right)^n$ and justify your reasoning. While this problem may be done fairly easily with “calculus 1” techniques, make sure you use only arguments we established (ask if unsure).

B6. Use the definition to show that $\lim_{x\to c}(x^3 + 3x^2 - 5x + 1) = c^3 + 3c^2 - 5c + 1$ for any $c \in \mathbf{R}$.

B7. Suppose $A \subseteq \mathbf{R}$ is a bounded set. If $\sup A \notin A$, show $\sup A$ is a cluster point of A. What can you say if $\sup A \in A$?

Type C problems (12pts each)

C1. Show that the set of all functions $\mathbf{N} \to \{0, 1\}$ is uncountable. *Hint: any function* $a : \mathbf{N} \to \{0, 1\}$ *may be thought of as a sequence of 0's and 1's.*

C2. Let the sequence x_n be recursively given by: $x_1 > 0$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$.
a) Show for any $a > 0$: $\frac{1}{2}\left(a + \frac{5}{a}\right) \geq \sqrt{5}$.
b) Show that x_n is a contractive sequence (part a) will help).
c) Find $\lim x_n$ if it exists.