Advanced Calculus 1 — Exam 3
MAT 525/625, Fall 2014 — D. Ivanšić

Name: ____________________

Show all your work!

Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define a cluster point of a set $A \subseteq \mathbf{R}$.

Theory 2. (3pts) Define what $\lim_{x\to\infty} f(x) = \infty$ means.

Theory 3. (3pts) State sequential criterion for convergence for the case $\lim_{x\to c} f(x) = L$.

Type A problems (8pts each)

A1. Find the limits, if they exist (just a computation is expected): a) $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$ b) $\lim_{x\to\infty} \frac{x^2-7x+12}{2x^2+5x+4}$.

A2. Does $\lim_{x\to 0} x \sin^2\left(\frac{1}{x}\right)$ exist? If yes, find it, if not, justify.

A3. Let $f : \mathbf{R} \to \mathbf{R}$ be the function given by: $f(x) = x$, if x is rational, $f(x) = x^2$ if x is irrational. find $\lim_{x\to 0} f(x)$ if it exists.

A4. Prove the limit theorem by definition: if $\lim_{x\to-\infty} f(x) = L$ and $\lim_{x\to-\infty} g(x) = M$, then $\lim_{x\to-\infty} (f(x) + g(x)) = L + M$.

A5. Use the definition to show that $\lim_{x\to\infty} \frac{2x-4}{5x+2} = \frac{2}{5}$.

Type B problems (8pts each)

B1. Use the definition to show that $\lim_{x\to c}(x^3 + 2x - 7) = c^3 + 2c - 7$ for any $c \in \mathbf{R}$.

B2. Prove the extended limit law $L \cdot \infty = \infty$, where $L > 0$, that is, if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = \infty$, show that $\lim_{x\to c}(f(x)g(x)) = \infty$.

B3. Suppose $\lim_{x\to 0+} xf(x) = L$, $L > 0$. Show that $\lim_{x\to 0+} f(x) = \infty$.

B4. Suppose $A \subseteq \mathbf{R}$ is a bounded set. If $\sup A \notin A$, show $\sup A$ is a cluster point of A. What can you say if $\sup A \in A$?

B5. Suppose $\lim_{x\to\infty} f(x) = \infty$ and that $g(x)$ is bounded on some interval (a, ∞). Show that $\lim_{x\to\infty} (f(x) - g(x)) = \infty$.

Type C problems (12pts each)

C1. Suppose $f, g : \mathbf{R} \to \mathbf{R}$, and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to L} g(x) = M$. Contrary to what one may think, $\lim_{x\to c} g(f(x)) = M$ is not always true. Give an example where it is not. (This one is subtle and hinges on the possibility that $g(L) \neq M$.)