**Advanced Calculus 1 — Exam 2**
**MAT 525/625, Fall 2014 — D. Ivanšić**

**Name:** ____________________

*Show all your work!*

*Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.*

**Theory 1.** (3pts) Define a subsequence of a sequence.

**Theory 2.** (3pts) Define when a sequence tends to $\infty$.

**Theory 3.** (3pts) State the Monotone Convergence Theorem.

## Type A problems (5pts each)

**A1.** Use the definition of the limit to show $\lim \frac{2n+3}{n-7} = 2$.

**A2.** Find $\lim \sqrt[2n]{7n^3}$.

**A3.** If $0 < a < b$, find $\lim(a^n + b^n)^{\frac{1}{n}}$.

**A4.** Establish convergence or divergence of $x_n = \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{2n}}$.

**A5.** If $\lim x_n = \infty$ and $\lim y_n = \infty$, prove using the definition that $\lim(x_n + y_n) = \infty$.

**A6.** Show $\lim \left(1 + \frac{1}{n}\right)^{n^2} = \infty$. Make sure you use only arguments we established.

## Type B problems (8pts each)

**B1.** Use the squeeze theorem to determine $\lim \sqrt[n]{n^2 - 3n + 1}$.

**B2.** Let the sequence $x_n$ be recursively given by: $x_1 = 1$, $x_{n+1} = \sqrt{7 + 2x_n}$. Show that this sequence converges and find its limit.

**B3.** Find $\lim \left(1 + \frac{1}{n^2}\right)^n$ and justify your reasoning. While this problem may be done fairly easily with “calculus 1” techniques, make sure you use only arguments we established (ask if unsure).

**B4.** Show that if a sequence $(x_n)$ is unbounded, then there exists a subsequence $(x_{n_k})$ such that $\lim \frac{1}{x_{n_k}} = 0$.

**B5.** Use the definition to show that the sequence $x_n = \frac{n^2}{n^2+1}$ is Cauchy.

**B6.** Let the sequence $x_n$ be recursively given by: $|x_1| \leq 1$, $x_{n+1} = \frac{1}{5}(x_n^3 + x_n - 1)$. Show that this sequence is contractive and write the equation that its limit satisfies (do not solve the equation, since it is not easy).

## Type C problems (12pts each)

**C1.** Let the sequence $x_n$ be recursively given by: $x_1 > 0$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$.
a) Show for any $a > 0$: $\frac{1}{2}\left(a + \frac{5}{a}\right) \geq \sqrt{5}$.
b) Show that $x_n$ is a contractive sequence (part a) will help).
c) Find $\lim x_n$ if it exists.

**C2.** Inspired by a student question, this problem is a negative version of the homework problem: if the even terms and the odd terms of a sequence converge to the same limit, then the sequence converges to this limit. We give an example of a sequence $(x_n)$ that can be broken up into infinitely many "disjoint" subsequences, all of which converge to the same limit, but $(x_n)$ itself diverges.

For every prime $p$ build the set $A_p \subseteq \mathbf{N}$ inductively as follows:

$A_2 = \{2, 4, 6, \dots\}$, all multiples of 2.
$A_3 = \{3, 6, 9, \dots\} \setminus A_2$,
$A_5 = \{5, 10, 15, \dots\} \setminus (A_2 \cup A_3)$, and so on: if $p$ is the first prime after $q$, then
$A_p = \{p, 2p, 3p, \dots\} \setminus (A_2 \cup A_3 \cup \cdots \cup A_q)$.

a) Show that $A_p$ is infinite.
b) Show that $A_p \cap A_q = \emptyset$ if $p \neq q$ and $\mathbf{N} = A_2 \cup A_3 \cup A_5 \cup \dots$.

Define the sequence: $x_n = \begin{cases} 1, & \text{if } n \text{ is prime} \\ 0, & \text{if } n \text{ is not prime.} \end{cases}$

Now suppose the infinitely many elements of $A_p$ are labeled in increasing order as $A_p = \{n_{p1}, n_{p2}, \dots, n_{pk}, \dots\}$. For every prime $p$, consider the subsequence $X_p$ of $(x_n)$ whose $k$-th term is $x_{n_{pk}}$ (we may write this as $X_{pk} = x_{n_{pk}}$).

c) Show that $x_n$ diverges.
d) Show that all the sequences $X_p$ converge to the same limit.