Advanced Calculus 1 — Exam 1 **MAT 525/625, Fall 2014 — D. Ivanšić**	**Name:** ____________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define the ϵ-neighborhood of a.

Theory 2. (3pts) Define the supremum of a set.

Theory 3. (3pts) If S and T are sets and $T \subseteq S$, state the theorem (2 claims) that relates countability of S to countability of T.

Type A problems (5pts each)

A1. Show using Mathematical Induction that $5^n - 4n - 1$ is divisible by 16 for all $n \in \mathbf{N}$.

A2. Let A be countable and B uncountable. Prove that $A \cup B$ is uncountable and $A \cap B$ is countable.

A3. If $a \in \mathbf{R}$, use algebraic properties of $\mathbf{R}$ to show that $-(-a) = a$, and $1/(-a) = -(1/a)$ (assuming $a \neq 0$).

A4. Show that $x + \frac{1}{x} \geq 2$ for all $x > 0$ and that equality holds if and only if $x = 1$.

A5. If they exist, find a lower bound of S, an upper bound of S, $\inf S$ and $\sup S$. There is no need to justify. a) $S = (-1, 4]$ b) $S = \left\{ \frac{1}{\sqrt{n}} \mid n \in \mathbf{N} \right\}$

Type B problems (8pts each)

B1. Let A be a denumerable collection of lines in the plane. Show that the set of all points that are intersections of any two of those lines is countable.

B2. Let $a \in \mathbf{N}$ be an even number that is not divisible by 4. Show that $\sqrt{a}$ is not a rational number.

B3. Determine and sketch the set of points in the plane satisfying $2|x| + |y| \leq 3$.

B4. Let $S = (1, \infty)$. If they exist, find a lower bound of S, an upper bound of S, $\inf S$ and $\sup S$. Prove the details, including nonexistence of any of the quantities.

B5. Let S be a nonempty set that is bounded above. Show that $\sup e^S = e^{\sup S}$. (As expected, $e^S = \{e^s \mid s \in S\}$.)

Type C problems (12pts each)

C1. Let $S = \{x \in \mathbf{R} \mid x^2 + bx + c = 0 \text{ for some } b, c \in \mathbf{Z}\}$. That is, S is the set of all solutions of quadratic equations $x^2 + bx + c = 0$ with coefficients $b, c \in \mathbf{Z}$. Show that S is countable.

C2. Show that the set of all functions $\mathbf{N} \to \{0, 1\}$ is uncountable. *Hint: any function* $a : \mathbf{N} \to \{0, 1\}$ *may be thought of as a sequence of 0's and 1's.*

C3. Let $f : \mathbf{R} \to \mathbf{R}$ be a strictly increasing function, and S a nonempty set that is bounded above. Although $\sup f(S) = f(\sup S)$ is true for reasonable functions f, find a function f — obviously, unreasonable — for which this is false.