Advanced Calculus 1 — Problem Farm 1 **MAT 525/625 Fall 2019 — D. Ivanšić**

Sets and Functions

Type A problems (5pts each)

A1. Give an example of a function $f : \mathbf{Z} \to \mathbf{Z}$ that is a bijection (but not the identity).

A2. Give an example of a function $f : \mathbf{N} \to \mathbf{N}$ that is a bijection (but not the identity).

A3. Give an example of a function $f : \mathbf{N} \to \mathbf{N}$ that is injective, but not surjective.

A4. Give an example of a function $f : \mathbf{Z} \to \mathbf{Z}$ that is injective, but not surjective.

A5. Give an example of a function $f : \mathbf{N} \to \mathbf{N}$ that is surjective, but not injective.

A6. Give an example of a function $f : \mathbf{Z} \to \mathbf{Z}$ that is surjective, but not injective.

A7. Give an example of a function $f : \mathbf{R} \to \mathbf{R}$ that is surjective, but not injective.

Type B problems (8pts each)

B1. Give an example of a function $f : A \to B$, where both A and B are finite sets (i.e., you can draw a picture with blobs), and sets $E, F \subset A$, for which $f(E \cap F) \neq f(E) \cap f(F)$.

B2. Let $p : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be given by $p(x, y) = x$.
a) Is p injective? Surjective?
b) Find $p^{-1}([-1, 5))$, and draw it in the plane.
c) Find $p^{-1}(\mathbf{Q})$, and draw it in the plane.

B3. Let $p : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be given by $p(x, y) = 2x + y$.
a) Is p injective? Surjective?
b) Find $p^{-1}([-1, 5))$, and draw it in the plane.
c) Find $p^{-1}(\mathbf{Q})$, and draw it in the plane.

B4. Let $p : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be given by $p(x, y) = 3x - 2y + 4z$.
Find $p^{-1}((-\infty, 4])$, and draw it in $\mathbf{R}^3$.

Type C problems (12pts each)

(none)

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Real Numbers

Type A problems (5pts each)

For the sets listed below do the following (justify each answer a little, perhaps with a drawing, but do not go into an axiom-based proof):
a) Find an upper bound and a lower bound, if any.
b) Find $\inf S$ and $\sup S$, if they exist.

A1. $S = \{1, 3, 7, 10\}$

A2. $S = (-\infty, 5]$ **A3.** $S = (-1, 4)$ (open interval) **A4.** $S = \{x \in \mathbf{Q} \mid 3 \le x < 7\}$

A5. $S = \left\{\frac{1}{n} \mid n \in \mathbf{N}\right\}$ **A6.** $S = \left\{(-1)^n \frac{1}{n} \mid n \in \mathbf{N}\right\}$

A7. $S = \{e^x \mid x \in \mathbf{R}\}$ **A8.** $S = \{\arctan x \mid x \in \mathbf{R}\}$

A9. $S = \left\{(-1)^n \left(2 - \frac{1}{n}\right), \mid n \in \mathbf{N}\right\}$ **A10.** $S = \{x \in \mathbf{Q} \mid x^2 < 2\}$

Type B problems (8pts each)

B1. Show that the set of complex numbers $\mathbf{C}$ cannot be made into an ordered field. In other words, show that it is impossible to find a set $P \subset \mathbf{C}$ that satisfies the order properties from 2.1.5. (Hint: suppose it is possible and derive a contradiction using the element $i \in \mathbf{C}$.)

B2. Formulate and prove statements for $\inf S$ that are analogous to Lemmas 2.3.3 and 2.3.4.

B3. Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$.
a) Show that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.
b) Use a) and induction to prove the binomial formula: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.

Type C problems (12pts each)

C1. (Juicy!) Do problem 2.1.9 for the set $K = \{s + t\sqrt[3]{2} + u\sqrt[3]{4} \mid s, t, u \in \mathbf{Q}\}$. Or, if you prefer, do it more generally for the set $K = \{s + t\alpha + u\alpha^2 \mid s, t, u \in \mathbf{Q}\}$, where $\alpha \notin \mathbf{Q}$ is a number for which $\alpha^3 \in \mathbf{Q}$. (Hint for part b of 2.1.9: you are trying to rationalize the denominator in $\frac{1}{s+t\alpha+u\alpha^2}$. If you multiply numerator and denominator by $x + y\alpha + z\alpha^2$, what conditions on x, y, z give you a rational denominator, and can you always satisfy the conditions?)

C2. Show for every $n \ge 3$: $\sqrt[n+1]{n+1} < \sqrt[n]{n}$. (Hint: show it is equivalent to $(n+1)^n < n^{n+1}$, and prove this, perhaps with the help of the binomial formula.)