Calculus 1 — Final Exam MAT 250, Fall 2019 — D. Ivanšić	Name: Saul Ocean

Show all your work!

1. (15pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

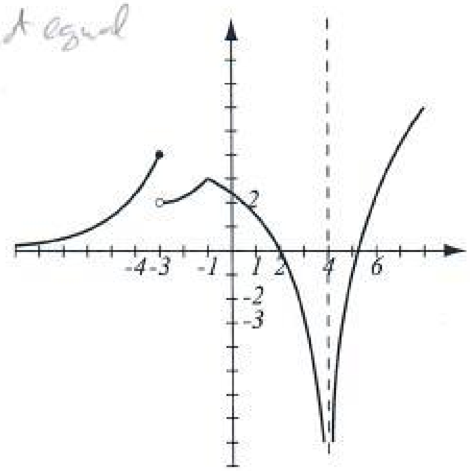
$\lim_{x\to -3^-} f(x) = 4$

$\lim_{x\to -3^+} f(x) = 2$

$\lim_{x\to -3} f(x) =$ does not exist, one-sided limits not equal

$\lim_{x\to 4} f(x) = -\infty$

$\lim_{x\to \infty} f(x) = \infty$



List points where f is not continuous and explain why.

$x=-3$, $\lim_{x\to -3} f(x)$ d.n.e.

List points where f is not differentiable and explain why.

$x=-3$ not continuous there, so not differentiable

$x=1$ sharp point

Find the following limits algebraically. Do not use L'Hospital's rule.

2. (6pts) $\lim_{x\to 2^+} \frac{x^2+3}{2-x} = \frac{7}{0-} = -\infty$

$y=2-x$

$+$ 2 $-$

when $x>2$

$2-x<0$

3. (5pts) $\lim_{x\to -4} \frac{x^2-x-20}{x^2-16} = \lim_{x\to -4} \frac{(x-5)\cancel{(x+4)}}{(x-4)\cancel{(x+4)}} = \frac{-4-5}{-4-4} = \frac{9}{8}$

4. (6pts) The equation $x^2 + 2 = 2^x$ is given. Use the Intermediate Value Theorem to show it has a solution.

$\underbrace{2^x - x^2}_{f(x),\ \text{continuous}} = 2$

$f(0) = 1 - 0 = 1$

$f(5) = 32 - 25 = 7$

Since $f(0) < 2 < f(5)$, by IVT there is a number c in $(0,5)$ such that $f(c) = 2$, in other words, equation has a solution.

5. (9pts) Find the equation of the tangent line to the curve $y = x^2 + \dfrac{1}{x}$ at the point where $x = 2$.

$y' = 2x - \frac{1}{x^2}$

$y'(2) = 4 - \frac{1}{4} = \frac{15}{4}$

$y(2) = 4 + \frac{1}{2} = \frac{9}{2}$

$y - \frac{9}{2} = \frac{15}{4}(x-2)$

$y - \frac{9}{2} = \frac{15}{4}x - \frac{15}{2}$

$y = \frac{15}{4}x - \frac{6}{2}$

$y = \frac{15}{4}x - 3$

6. (10pts) Let $f(x) = \sqrt{x}$.

a) Write the linearization of $f(x)$ at $a = 25$.

b) Use the linearization to estimate $\sqrt{27}$ and compare to the calculator value of 5.196152.

a) $L(x) = f(a) + f'(a)(x-a)$

$f(x) = \sqrt{x}$ $\quad f(25) = 5$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f'(25) = \frac{1}{10}$

$L(x) = 5 + \frac{1}{10}(x - 25)$

b) $L(27) = 5 + \frac{1}{10}\cdot(27-25)$

$= 5 + \frac{2}{10} = 5.2$, quite close to 5.196152

7. (26pts) Let $f(x) = \dfrac{x^2}{e^x}$. Draw an accurate graph of f by following the guidelines.

a) Find the intervals of increase and decrease, and local extremes.

b) Find the intervals of concavity and points of inflection.

c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. Use L'Hospital's rule where necessary.

d) Use information from a)–c) to sketch the graph.

$$f'(x) = \frac{2xe^x - x^2e^x}{(e^x)^2} = \frac{\cancel{e^x}(2x-x^2)}{(e^x)^{\cancel{2}}} = \frac{2x-x^2}{e^x}$$

$$f''(x) = \frac{(2-2x)e^x - (2x-x^2)e^x}{(e^x)^2} = \frac{(x^2-4x+2)\cancel{e^x}}{(e^x)^{\cancel{2}}} = \frac{x^2-4x+2}{e^x}$$

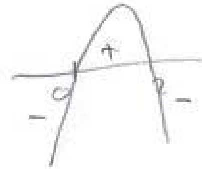
a) $f'(x)=0$ (f' always exists)

$2x - x^2 = 0$

$x(2-x) = 0$

$x = 0, 2$

Since $e^x > 0$, f' has sign of $2x - x^2$



	0		2	
f' −	0	+	0	−
f ↘	loc min	↗	loc max	↘

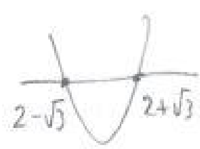
b) $f''(x) = 0$ (f'' always exists)

$x^2 - 4x + 2 = 0$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\cdot 1\cdot 2}}{2\cdot 1} = \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

Since $e^x > 0$, f'' has sign of $x^2 - 4x + 2$

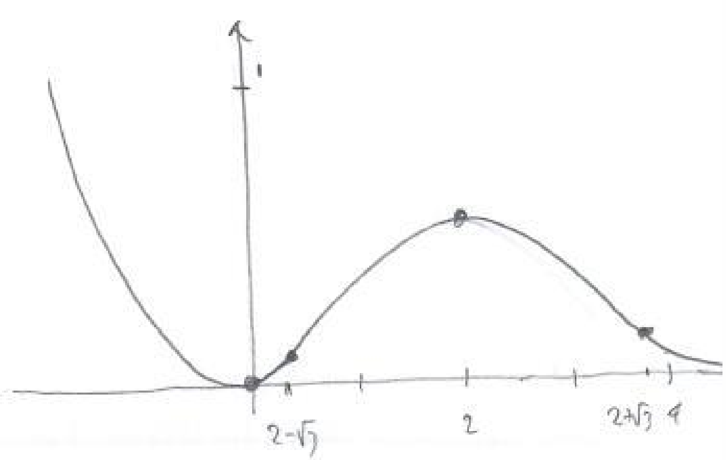


	$2-\sqrt{3}$		$2+\sqrt{3}$	
f'' +	0	−	0	+
f CU	IP	CD	IP	CU

c) $$\lim_{x\to\infty} \frac{x^2 \;(\to\infty)}{e^x \;(\to\infty)} \overset{L'H}{=} \lim_{x\to\infty} \frac{2x \;(\to\infty)}{e^x \;(\to\infty)} \overset{L'H}{=} \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$$

$$\lim_{x\to-\infty} \frac{x^2}{e^x} = \lim_{x\to-\infty} \frac{\infty}{0+} = \infty\cdot\frac{1}{0+} = \infty\cdot\infty = \infty$$

x	$\frac{x^2}{e^x}$
0	0
2	$\frac{4}{e^2} \approx \frac{4}{9}$
$2-\sqrt{3}$	$\frac{(2-\sqrt{3})^2}{e^{2-\sqrt{3}}}$ difficult, >0
$2+\sqrt{3}$	$\frac{(2+\sqrt{3})^2}{e^{2+\sqrt{3}}}$ difficult, >0



8. (10pts) Let $f(x) = x + 2\cos x$. Find the absolute minimum and maximum values of f on the interval $\left[0, \frac{\pi}{2}\right]$.

$f'(x) = 1 - 2\sin x$

Critical pt:

$1 - 2\sin x = 0$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ ← not in $\left[0, \frac{\pi}{2}\right]$

x	$x + 2\cos x$
$\frac{\pi}{6}$	$\frac{\pi}{6} + 2\cdot\frac{\sqrt{3}}{2} \approx \frac{1}{2} + 1.7 = 2.25$ abs. max
0	$0 + 2 = 2$
$\frac{\pi}{2}$	$\frac{\pi}{2} + 0 \simeq 1.5$ abs. min

9. (6pts) Find $f(x)$ if $f'(x) = \sqrt[3]{x} + \sec x \tan x$ and $f(0) = 3$.

$f'(x) = x^{\frac{1}{3}} + \sec x \tan x$

$f(x) = \dfrac{x^{\frac{4}{3}}}{\frac{4}{3}} + \sec x + C$

$3 = f(0) = \frac{3}{4} 0^{\frac{4}{3}} + \frac{1}{\cos 0} + C$

$3 = 1 + C$

$C = 2$

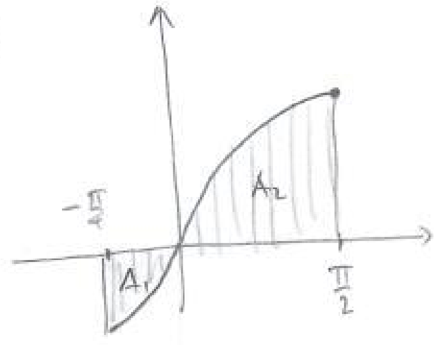
$f(x) = \frac{3}{4} x^{\frac{4}{3}} + \sin x + 2$

10. (11pts) Consider the integral $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\theta \, d\theta$.

a) Use a picture and the "area" interpretation of the integral to determine whether this integral is positive or negative.

b) Use the Evaluation Theorem to find the integral and verify your conclusion from a).

a)



$\displaystyle\int_{-\pi/4}^{\pi/2} \sin\theta \, d\theta = -A_1 + A_2 = A_2 - A_1 > 0$

since clearly $A_2 > A_1$

b) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\theta \, d\theta = -\cos\theta \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= -\left(\cos\frac{\pi}{2} - \cos\left(-\frac{\pi}{4}\right)\right)$

$= -\left(0 - \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} > 0.$

Use the substitution rule in the following integrals:

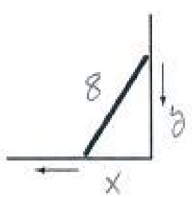
11. (7pts) $\displaystyle\int \frac{1}{(1+(\ln x)^2)\,x}\,dx =$ $\left[\begin{array}{l} u=\ln x \\ du=\frac{1}{x}dx \end{array}\right] = \int \frac{1}{1+u^2}\,du = \arctan u$

$= \arctan(\ln x) + C$

12. (9pts) $\displaystyle\int_{-1}^{1} \underbrace{(3x^2-4x)e^{x^3-2x^2+1}\,dx}_{du} =$ $\left[\begin{array}{ll} u = x^3-2x^2+1 & x=1,\ u=0 \\ du = 3x^2-4x\,dx & x=-1,\ u=-2 \end{array}\right]$

$= \displaystyle\int_{-2}^{0} e^u\,du = e^u\Big|_{-2}^{0} = e^0 - e^{-2} = 1 - \frac{1}{e^2}$

13. (12pts) An 8-foot ladder is sliding down the wall against which it is leaning. When the bottom of the ladder is 5 feet from the base of the wall, it is moving away from the wall at speed $\frac{1}{2}$ feet per second. How fast is the top of the ladder dropping at that moment?



Know: $x' = \frac{1}{2}$ ft/s

Need: y' when $x = 5$ ft

$x^2 + y^2 = 8^2 \quad \Big| \frac{d}{dt}$

$2xx' + 2yy' = 0$

$2yy' = -2xx'$

$y' = -\dfrac{2xx'}{2y} = -\dfrac{xx'}{y}$

When $x=5$, $5^2 + y^2 = 8^2$

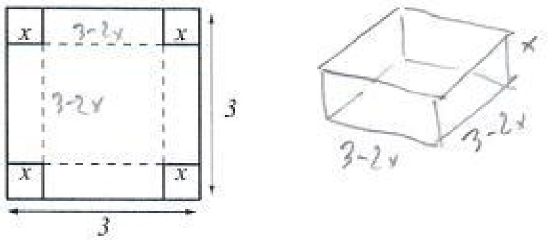
$y^2 = 64 - 25$

$y^2 = 39$

$y = \pm\sqrt{39} = \sqrt{39}$ since $y \geq 0$

$y' = -\dfrac{5\cdot\frac{1}{2}}{\sqrt{39}} = -\dfrac{5}{2\sqrt{39}}$ ft/s

14. (18pts) A 3×3 square piece of cardboard is to be made into a box by cutting out four smaller squares from the corners and folding the flaps upward. Dashed lines show where the cardboard is folded after the corner squares are removed. What size of removed squares produces the maximal possible volume of the resulting open-top box?



(Must have $x \geq 0$, $3-2x \geq 0$, $x \leq \frac{3}{2}$)

$V(x) = (3-2x)^2 \cdot x$

Job: maximize $V(x)$ on $[0, \frac{3}{2}]$

$V'(x) = 2(3-2x)(-2)x + (3-2x)^2 \cdot 1$

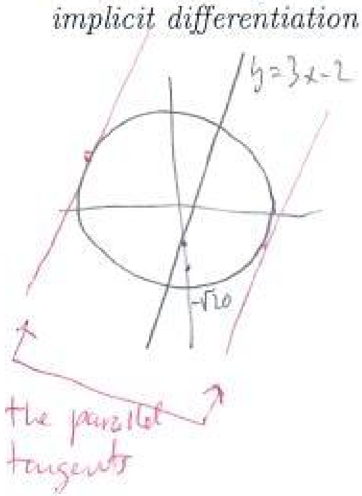
$= (3-2x)(-4x+3-2x)$

$= (3-2x)(3-6x)$

$V'(x)=0$ when $x = \frac{3}{2}$ or $x = \frac{1}{2}$

x	$(3-2x)^2 \cdot x$	
$\frac{1}{2}$	4	abs max.
0	0	
$\frac{3}{2}$	0	

Bonus. (10pts) Find points on the circle $x^2 + y^2 = 20$ where the tangent line is parallel to the line $6x - 2y = 4$. Draw the circle, the given line and the parallel tangent line(s). *(Hint: implicit differentiation is a little easier.)*



$6x - 2y = 4$

$y = \frac{-6x+4}{-2}$

$y = 3x - 2$

$x^2 + y^2 = 20 \;\Big|\; \frac{d}{dx}$

$2x + 2yy' = 0$

$y' = -\frac{2x}{2y} = -\frac{x}{y}$ = slope of tan. line

$\begin{cases} -\frac{x}{y} = 3 & \text{(slope is 3)} \\ x^2+y^2 = 20 & \text{(pt. is on circle)} \end{cases}$

$x = -3y$

$(-3y)^2 + y^2 = 20$

$10y^2 = 20$

$y^2 = 2$

$y = \pm\sqrt{2} \Rightarrow x = \mp 3\sqrt{2}$

The points where tangent line is parallel to given line are $(-3\sqrt{2}, \sqrt{2})$ and $(3\sqrt{2}, -\sqrt{2})$