Calculus 1 — Exam 5
MAT 250, Fall 2019 — D. Ivanšić

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Show all your work!

Find the following antiderivatives.

1. (3pts) $\int \sqrt[7]{x^3}\,dx = \int x^{\frac{3}{7}} = \frac{x^{\frac{10}{7}}}{\frac{10}{7}} = \frac{7}{10} x^{\frac{10}{7}} + C$

2. (3pts) $\int \frac{5}{\sqrt{1-x^2}}\,dx = 5 \arcsin x + C$

3. (3pts) $\int e^{4x+1}\,dx = \frac{e^{4x+1}}{4} + C$

4. (7pts) $\int \frac{s^3+s}{\sqrt{s}}\,ds = \int \frac{s^3+s}{s^{1/2}}\,ds = \int s^{\frac{5}{2}} + s^{\frac{1}{2}}\,ds = \frac{s^{\frac{7}{2}}}{\frac{7}{2}} + \frac{s^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{7} s^{\frac{7}{2}} + \frac{2}{3} s^{\frac{3}{2}} + C$

5. (7pts) Find $f(x)$ if $f'(x) = \frac{3}{1+x^2} + \frac{1}{x}$ and $f(1) = 5$.

$f(x) = 3\arctan x + \ln x + C$

$5 = f(1) = 3\arctan x + \underbrace{\ln 1}_{=0} + C$

$5 = 3 \cdot \frac{\pi}{4} + C$

$C = 5 - \frac{3\pi}{4}$

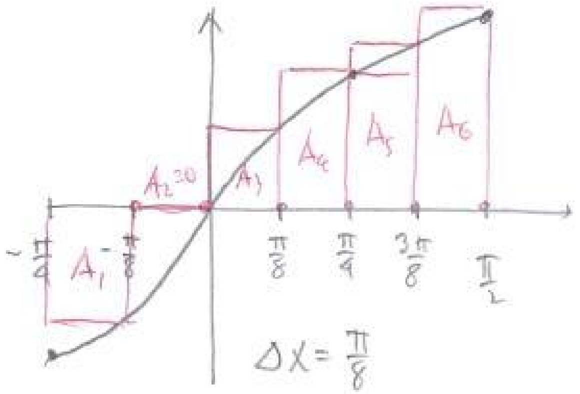
$f(x) = \arctan x + 5 - \frac{3\pi}{4}$

6. (6pts) Write using sigma notation:

$\underbrace{\frac{5}{6} + \frac{7}{8} + \frac{9}{10}}_{\text{evens, from } 2\cdot 3 \text{ to } 2\cdot 9} + \cdots + \frac{17}{18} = \sum_{i=3}^{9} \frac{2i-1}{2i}$

one less than denominator

7. (15pts) The function $f(x) = \sin x$ is given on the interval $[-\frac{\pi}{4}, \frac{\pi}{2}]$.
a) Write the Riemann sum R_6 for this function with six subintervals, taking sample points to be right endpoints. Do not evaluate the expression.
b) Illustrate with a diagram, where appropriate rectangles are clearly visible. What does R_6 represent?



a) $\sum_{i=1}^{6} f(x_i)\Delta x$

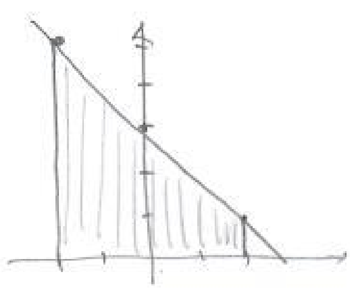
$= \frac{\pi}{8}\left(\sin(-\frac{\pi}{8}) + \sin 0 + \sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2}\right)$

b) Rieman sum represents

$-A_1 + A_2 + A_3 + A_4 + A_5 + A_6$,

where A_i is the area of the shown rectangles

8. (13pts) Find $\int_{-2}^{2} 3 - x\,dx$ in two ways (they'd better give you the same answer!):
a) Using the "area" interpretation of the integral. Draw a picture.
b) Using the Evaluation Theorem.



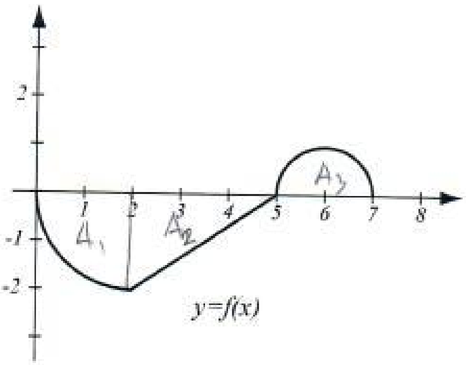
a) $\int_{-2}^{2} 3-x\,dx$ = shaded area (trapezoid)

$= \frac{1}{2}\cdot 4\cdot(1+5) = 12$

b) $\int_{-2}^{2} 3-x\,dx = 3x - \frac{x^2}{2}\Big|_{-2}^{2} = 3(2-(-2)) - \frac{1}{2}\underbrace{\left(2^2-(-2)^2\right)}_{=0}$

$= 12$

9. (7pts) The graph of a function f, consisting of lines and parts of circles, is shown. Evaluate the integrals.



$\int_2^5 f(x)\,dx = -A_2 = -\frac{1}{2}\,3\cdot 2 = -3$

$\int_5^7 f(x)\,dx = A_3 = \frac{1}{2}\cdot\pi\cdot 1^2 = \frac{\pi}{2}$

$\int_0^7 f(x)\,dx = -A_1 - A_2 + A_3 = -\frac{1}{4}\cdot 2^2\cdot\pi - 3 + \frac{\pi}{2}$

$= -3 - \frac{\pi}{2}$

Use the substitution rule in the following integrals:

10. (8pts) $\displaystyle\int \frac{3x^2+4x}{(x^3+2x^2+7)^2}\,dx = \begin{bmatrix} u = x^3+2x^2+7 \\ du = 3x^2+4x\,dx \end{bmatrix} = \int \frac{du}{u^2} = \int u^{-2}\,du$

$= \frac{u^{-1}}{-1} = -\frac{1}{u} = -\frac{1}{x^3+2x^2+7} + C$

11. (10pts) $\displaystyle\int_0^{\ln 5} \frac{e^x}{\sqrt{4+e^x}}\,dx = \begin{bmatrix} u = 4+e^x & x = \ln 5, \; u = e^{\ln 5}+4 = 9 \\ du = e^x dx & x = 0 \quad u = e^0 + 4 = 5 \end{bmatrix}$

$= \displaystyle\int_5^9 \frac{1}{\sqrt{u}}\,du = \int_5^9 u^{-\frac{1}{2}}\,du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}}\Big|_5^9 = 2\sqrt{u}\Big|_5^9 = 2(\sqrt{9}-\sqrt{5}) = 2(3-\sqrt{5})$

12. (8pts) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (\sin^2 x - 3\sin x + 3)\cos x\,dx = \begin{bmatrix} u = \sin x & x = \frac{5\pi}{6} \quad u = \sin\frac{5\pi}{6} = \frac{1}{2} \\ du = \cos x\,dx & x = \frac{\pi}{6} \quad u = \sin\frac{\pi}{6} = \frac{1}{2} \end{bmatrix}$

$= \displaystyle\int_{1/2}^{1/2} u^2 - 3u + 3\,du = 0$ (same bound top and bottom)

13. (10pts) A ball traveling upwards has speed $v(t) = 27 - 10t$ meters per second..
a) Use the Net Change Theorem to find by how much the height of the ball has changed from $t = 0$ to $t = 3$.
b) If at time $t = 0$ the ball was at height 18 meters, at what height is it at $t = 3$?

a) $v(t) = s'(t)$

$$s(3) - s(0) = \int_0^3 s'(t)\,dt = \int_0^3 v(t)\,dt = \int_0^3 27 - 10t\,dt$$

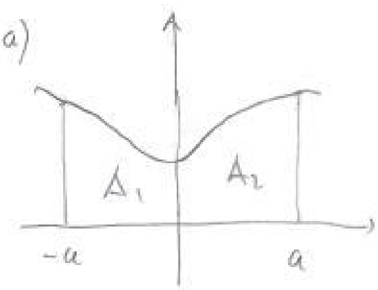
$$= \left(27t - 10\frac{t^2}{2}\right)\Big|_0^3 = \left(27t - 5t^2\right)\Big|_0^3 = 27(3-0) - 5(9-0)$$

$$= 81 - 45 = 36 \text{ meters}$$

b) $s(3) = s(0) + s(3) - s(0) = 18 + 36 = 54$

Bonus. (10pts) Justify the following statements with pictures.
a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$
b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

a)

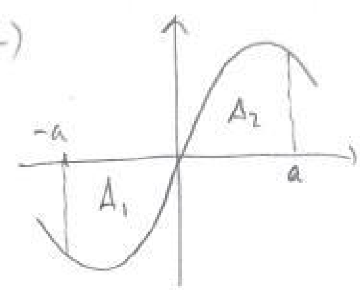


An even function is symmetric wrt y-axis

$$\int_{-a}^{a} f = A_1 + A_2 = A_2 + A_2 = 2A_2 = 2\int_0^a f$$

$A_1 = A_2$ due to symmetry

b)



An odd function is symmetric wrt. origin

$$\int_{-a}^{a} f = -A_1 + A_2 = -A_2 + A_2 = 0$$

$A_1 = A_2$ due to symmetry