Calculus 1 — Exam 4	**Name:** Saul Ocean
MAT 250, Fall 2019 — D. Ivanšić	*Show all your work!*

1. (30pts) Let $f(x) = x^3e^x$. Draw an accurate graph of f by following the guidelines.
a) Find the intervals of increase and decrease, and local extremes.
b) Find the intervals of concavity and points of inflection.
c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
d) Use information from a)–c) to sketch the graph.

$$f'(x) = 3x^2e^x + x^3e^x$$
$$= (x^3+3x^2)e^x = x^2(x+3)e^x$$
$$f''(x) = (3x^2+6x)e^x + (x^3+3x^2)e^x$$
$$= (x^3+6x^2+6x) = x(x^2+6x+6)e^x$$

c) $\lim\limits_{x\to\infty} x^3e^x = \infty\cdot\infty = \infty$

$\lim\limits_{x\to-\infty} x^3e^x = [-\infty\cdot 0] = \lim\limits_{x\to-\infty} \dfrac{x^3}{e^{-x}}$

$\overset{L'H}{=} \lim\limits_{x\to-\infty} \dfrac{3x^2}{-e^{-x}} \overset{L'H}{=} \lim\limits_{x\to-\infty} \dfrac{6x}{e^{-x}} \overset{L'H}{=} \lim\limits_{x\to-\infty} \dfrac{6}{-e^{-x}} = \dfrac{6}{-\infty} = 0$

a) Crit. pts: $x = 0, -3$ $(e^x > 0,\ x^2 \geq 0)$
Sign of f' only depends on $x+3$

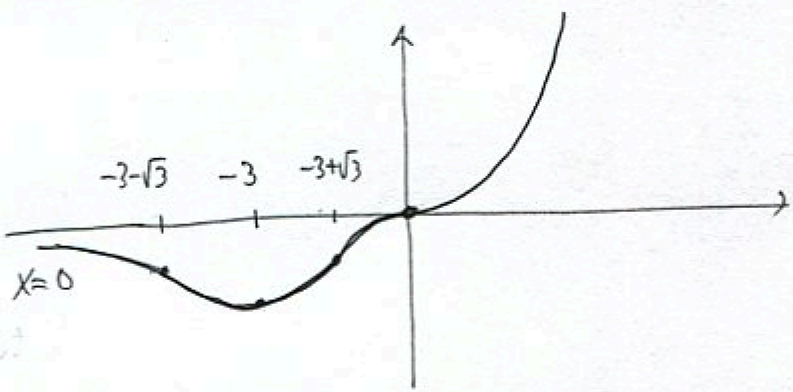
$x+3$: $-$ before -3, $+$ after

	-3		0	
f'	$-$ 0	$+$	0	$+$
f	$\searrow$ loc. min	$\nearrow$		$\nearrow$

d)

x	x^3e^x
-3	$-27e^{-3}$
0	0
$-3-\sqrt3$	$(-3-\sqrt3)^3e^{-3-\sqrt3} = ?$
$-3+\sqrt3$	$(-3+\sqrt3)^3e^{-3+\sqrt3} = ?$

$\}$ negative



b) 2nd order critical points:
$x^2+6x+6=0$
$$x = \frac{-6 \pm \sqrt{6^2-4\cdot6}}{2\cdot1} = \frac{-6\pm\sqrt{12}}{2}$$
$$= \frac{-6\pm2\sqrt3}{2} = -3\pm\sqrt3$$

Sign depends on x, x^2+6x+6

	$-3+\sqrt3$	$-3+\sqrt3$	0	
x	$-$	$-$	$-$ 0	$+$
x^2+6x+6	$+$ 0	$-$ 0	$+$	$+$
f''	$-$ 0	$+$ 0	$-$ 0	$+$
f	CD infl	CU infl	CD infl	CU

2. (18pts) Let $f(x) = \sin^3 x + \cos^3 x$. Find the absolute minimum and maximum values of f on the interval $[0, \pi]$.

$$f'(x) = 3\sin^2 x \cos x + 3\cos^2 x(-\sin x)$$
$$= 3\sin x \cos x\,(\sin x - \cos x)$$

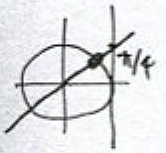
$\sin x = 0, \quad x = 0, \pi$

$\cos x = 0, \quad x = \frac{\pi}{2}$

$\sin x - \cos x = 0$

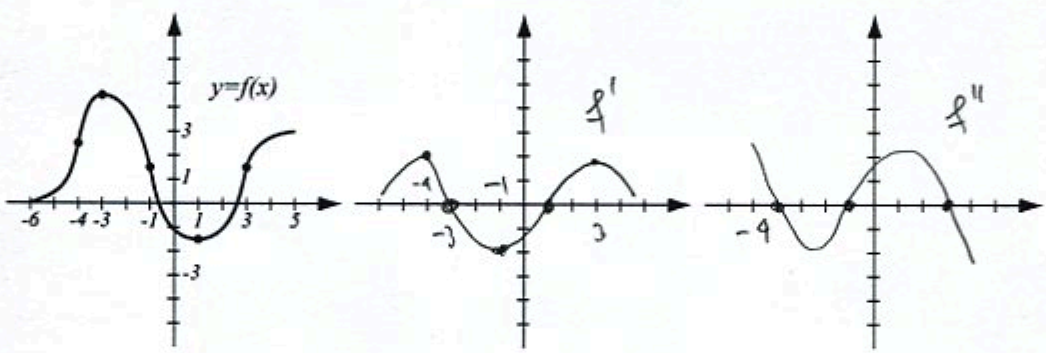
$\sin x = \cos x$

$\tan x = 1 \qquad x = \frac{\pi}{4}$



x	$\sin^3 x + \cos^3 x$	
0	$0 + 1 = 1$	abs max at $x=0$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}\right)^3 + \left(\frac{\sqrt{2}}{2}\right)^3 = \frac{2\sqrt{2}}{8}\cdot 2 = \frac{\sqrt{2}}{2}$	
$\frac{\pi}{2}$	$1 + 0 = 1$	abs max at $x = \frac{\pi}{2}$
π	$0 - 1 = -1$	abs min at π

3. (14pts) The graph of f is given. Use it to draw the graphs of f' and f'' in the coordinate systems provided. Pay attention to increasingness, decreasingness and concavity of f. The relevant special points have been highlighted.



$f' \gtrless 0 \iff f$ incr/decr.

f' incr/decr $\iff f$ CU/CD

4. (16pts) Consider $f(x) = \dfrac{x+5}{x+1}$ on the interval $[0, 3]$.

a) Verify that the function satisfies the assumptions of the Mean Value Theorem.

b) Find all numbers c that satisfy the conclusion of the Mean Value Theorem.

a) f is continuous on $[0,3]$ and differentiable on $(0,3)$
(a rational function, defined on $[0,3]$)

b) $\dfrac{f(3)-f(0)}{3-0} = \dfrac{\frac{8}{4}-\frac{5}{1}}{3} = \dfrac{2-5}{3} = -1$

Solve $-\dfrac{4}{(x+1)^2} = -1$

$f'(x) = \dfrac{1\cdot(x+1) - 1\cdot(x+5)}{(x+1)^2} = \dfrac{-4}{(x+1)^2}$

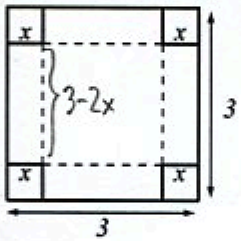
$(x+1)^2 = 4$

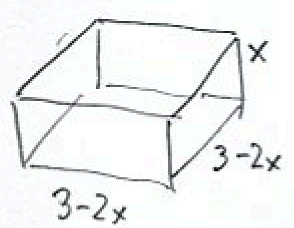
$x+1 = \pm 2$

$x = -1 \pm 2 = -3, 1$

1 is in the interval $(0,3)$.

5. (22pts) A 3×3 square piece of cardboard is to be made into a box by cutting out four smaller squares from the corners and folding the flaps upward. Dashed lines show where the cardboard is folded after the corner squares are removed. What size of removed squares produces the maximal possible volume of the resulting open-top box?





$V = (3-2x)(3-2x)\cdot x$

$= (3-2x)^2 \cdot x$

Must have $3-2x \geq 0$

$2x \leq 3$

$x \leq \frac{3}{2}$

Job: maximize $V(x)$ on $[0, \frac{3}{2}]$

$V' = 2(3-2x)(-2)\cdot x + (3-2x)^2 \cdot 1$

$= (3-2x)(-4x+3-2x)$

$= (3-2x)(3-6x)$

$3-2x=0 \qquad 3-6x=0$

$x = \frac{3}{2} \qquad x = \frac{1}{2}$

Closed interval method:

x	$V(x)$
0	0
$\frac{3}{2}$	0
$\frac{1}{2}$	$(3-1)^2 \cdot \frac{1}{2} = 2$ max volume

Bonus. (10pts) The Bernoulli inequality states that $(1+x)^n \geq 1+nx$ for every natural number n and every real number $x > -1$. Prove this inequality using calculus as follows:
a) Find the absolute minimum of the function $f(x) = (1+x)^n - nx$ on the interval $[-1, \infty)$.
b) Use the absolute minimum to conclude the inequality holds.

a) $f(x) = (1+x)^n - nx$

$f'(x) = n(1+x)^{n-1} - n = n\left((1+x)^{n-1} - 1\right)$

$(1+x)^{n-1} - 1 = 0$

$\underbrace{(1+x)}_{\geq 0}{}^{n-1} = 1$

$1+x = 1$

$x = 0$

	-1		0	
$f'(x)$		$-$	0	$+$
$f(x)$		$\searrow$	loc. min	$\nearrow$

When $-1<x<0$	$x>1$
$0<1+x<1$	$1+x>2$
so $(1+x)^{n-1}<1$	$(1+x)^{n-1}>2^{n-1}$
and $(1+x)^{n-1}-1<0$	$(1+x)^{n-1}-1>2^{n-1}-1 > 0$

Since f decreases up to $x=0$

and then increases, f has an abs. min at $x=0$

The abs. min value is $f(0) = 1$

b) Since $f(0)=1$ is the minimal value, $f(x) \geq 1$ for any x

in $[-1, \infty)$. Thus $(1+x)^n - nx \geq 1$

or $(1+x)^n \geq 1+nx$