| Calculus 1 — Exam 3 |
|---|
| MAT 250, Fall 2019 — D. Ivanšić |

**Name:** Saul Ocean

*Show all your work!*

Differentiate and simplify where appropriate:

**1.** (5pts) $\frac{d}{dx}\ln(3x^2-5x+2) = \frac{1}{3x^2-5x+2}\cdot(6x-5) = \frac{6x-5}{3x^2-5x+2}$

**2.** (6pts) $\frac{d}{dx}(x^{\frac{3}{2}}-4x^{\frac{1}{2}})e^x = \left(\frac{3}{2}x^{\frac{1}{2}}-4\cdot\frac{1}{2}x^{-\frac{1}{2}}\right)e^x+(x^{\frac{3}{2}}-4x^{\frac{1}{2}})e^x$

$= e^x\left(\frac{3}{2}x^{\frac{1}{2}}-2x^{-\frac{1}{2}}+x^{\frac{3}{2}}-4x^{\frac{1}{2}}\right) = e^x\left(x^{\frac{3}{2}}-\frac{5}{2}x^{\frac{1}{2}}-2x^{-\frac{1}{2}}\right)$

**3.** (6pts) $\frac{d}{du}\frac{\ln u}{u^2} = \frac{\frac{1}{u}\cdot u^2-\ln u\cdot 2u}{u^4} = \frac{u-2u\ln u}{u^4} = \frac{1-2\ln u}{u^3}$

**4.** (7pts) $\frac{d}{dx}\arctan\sqrt{x^2-1} = \frac{1}{1+(\sqrt{x^2-1})^2}\cdot\frac{1}{2\sqrt{x^2-1}}\cdot 2x = \frac{1}{1+x^2-1}\cdot\frac{x}{\sqrt{x^2-1}}$

$= \frac{x}{x^2\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}}$

**5.** (7pts) $\frac{d}{d\theta}\log_4\frac{1-\sec\theta}{1+\sec\theta} = \frac{d}{d\theta}\left(\log_4(1-\sec\theta)-\log_4(1+\sec\theta)\right)$

$= \frac{1}{(1-\sec\theta)\ln 4}(-\sec\theta\tan\theta) - \frac{1}{(1+\sec\theta)\ln 4}\cdot\sec\theta\tan\theta = \frac{-\sec\theta\tan\theta}{\ln 4}\left(\frac{1}{1-\sec\theta}+\frac{1}{1+\sec\theta}\right)$

$= -\frac{\sec\theta\tan\theta}{\ln 4}\,\frac{1+\sec\theta+1-\sec\theta}{(1-\sec\theta)(1+\sec\theta)} = \frac{-2\sec\theta\tan\theta}{(1-\sec^2\theta)\ln 4} = -\frac{2\sec\theta\tan\theta}{\ln 4\tan^2\theta} = -\frac{2\sec\theta}{\ln 4\cdot\tan\theta} = -\frac{2}{\ln 4\sin\theta}$

$= -\frac{2}{\ln 4}\csc\theta$

**6.** (10pts) Use logarithmic differentiation to find the derivative of $y=(x^2-1)^{x^2}$.

$y = (x^2-1)^{x^2}$

$\ln y = x^2\ln(x^2-1) \quad \Big| \frac{d}{dx}$

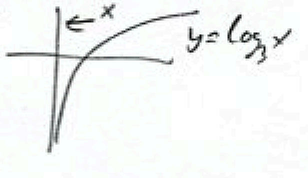
$\frac{1}{y}\cdot y' = 2x\ln(x^2-1)+x^2\cdot\frac{1}{x^2-1}\cdot 2x$

$y' = y\left(2x\ln(x^2-1)+\frac{2x^3}{x^2-1}\right)$

$y' = 2(x^2-1)^{x^2}\left(x\ln(x^2-1)+\frac{x^3}{x^2-1}\right)$

Find the limits algebraically. Graphs of basic functions will help, as will L'Hospital's rule, where appropriate.

**7.** (2pts) $\lim\limits_{x\to 0+} \log_3 x = -\infty$




**8.** (7pts) $\lim\limits_{x\to\infty} \arccos\left(\dfrac{x+4}{x^2-3x+12}\right) = \arccos\left(\lim\limits_{x\to\infty} \dfrac{x+4}{x^2-3x+12}\right)$

$$= \arccos\left(\lim_{x\to\infty} \frac{x\left(1+\frac{4}{x}\right)}{x^2\left(1-\frac{3}{x}+\frac{12}{x^2}\right)}\right) = \arccos\left(\lim_{x\to\infty} \frac{1}{x}\cdot\frac{1+\frac{4}{x}}{1-\frac{3}{x}+\frac{12}{x^2}}\right)$$

$$= \arccos\left(0\cdot\frac{1+0}{1-0+0}\right) = \arccos(0\cdot 1) = \arccos 0 = \frac{\pi}{2}$$

**9.** (6pts) $\lim\limits_{x\to0} \dfrac{\overset{1-1}{1-\cos x}}{\underset{0}{x^2}} \overset{L'H}{=} \lim\limits_{x\to0}\dfrac{\overset{\to 0}{-(-\sin x)}}{\underset{\to 0}{2x}} = \lim\limits_{x\to0}\dfrac{\cos x}{2} = \dfrac{1}{2}$

**10.** (9pts) $\lim\limits_{x\to\infty} \dfrac{\overset{\to\infty}{x^{\frac{3}{2}}}}{\underset{\to\infty}{1.1^x}} \overset{L'H}{=} \lim\limits_{x\to\infty}\dfrac{\overset{\to\infty}{\frac{3}{2}x^{\frac{1}{2}}}}{\underset{\to\infty}{1.1^x\cdot\ln 1.1}} \overset{L'H}{=} \lim\limits_{x\to\infty}\dfrac{\overset{\to 0}{\frac{3}{2}\cdot\frac{1}{2}x^{-\frac{1}{2}}}}{\underset{\to\infty}{1.1^x\cdot\ln1.1\cdot\ln1.1}} = \lim\limits_{x\to\infty}\dfrac{3}{4(\ln1.1)^2\cdot\sqrt{x}\,1.1^x}$

$$= \frac{3}{4(\ln1.1)^2\cdot\infty\cdot\infty} = \frac{3}{\infty} = 0$$

**11.** (8pts) $\lim\limits_{x\to0}(1-2x)^{\frac{1}{\sin x}} = e^{\lim\limits_{x\to0}\ln y} = e^{-2} = \dfrac{1}{e^2}$

$y = (1-2x)^{\frac{1}{\sin x}}$

$\lim\limits_{x\to0}\ln y = \lim\limits_{x\to0}\dfrac{\overset{\to \ln 1 = 0}{\ln(1-2x)}}{\underset{\to 0}{\sin x}} \overset{L'H}{=} \lim\limits_{x\to0}\dfrac{\frac{1}{1-2x}\cdot(-2)}{\cos x} = \dfrac{\frac{-2}{1-0}}{1} = -2$

$\ln y = \dfrac{1}{\sin x}\ln(1-2x)$

**12.** (10pts) Let $f(x) = \sqrt{x}$.
a) Write the linearization of $f(x)$ at $a = 16$.
b) Use the linearization to estimate $\sqrt{17}$ and compare to the calculator value of 4.123106.

a) $f'(x) = \frac{1}{2\sqrt{x}}$

$L(17) = \frac{17}{8} + 2 = \frac{33}{8} = 4\frac{1}{8} = 4.125$

$f(16) = 4$

$f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$

within $2\times10^{-3}$ of calculator value

b) $L(x) = f(16) + f'(16)(x-16)$

$= 4 + \frac{1}{8}(x-16)$

$= \frac{1}{8}x + 2$

**13.** (10pts) A cube is measured to have side length of 5 centimeters, with maximum error 2 millimeters. Use differentials to estimate the maximum possible error, the relative error and the percentage error when computing the volume of the cube.

$V = x^3$ $\quad$ $2\text{ mm} = 0.2\text{ cm}$

$dV = 3x^2 dx$

$dV = 3\cdot5^2\cdot0.2 = 75\cdot0.2 = 15\text{ cm}^3$

$\frac{dV}{V} = \frac{15}{5^3} = \frac{3\cdot\cancel{5}}{5^{\cancel{3}}} = \frac{3\cdot4}{25\cdot4} = \frac{12}{100} = 12\%$

**14.** (7pts) The table of values of $f(x)$ and $f'(x)$ is given at right. Use the theorem on derivatives of inverses to find $(f^{-1})'(1)$.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 4 | 1 | 0 | -1 |
| $f'(x)$ | -2 | -3 | -4 | -1 |

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(2)} = \frac{1}{-3} = -\frac{1}{3}$$

$f(x) = 1$

$x = 2$

from table

**Bonus.** (10pts) The function $f(x) = x^2 + 4x - 7$ is one-to-one on the domain $(-\infty, -2]$.
a) Use either the quadratic formula or completion of squares to find $f^{-1}(x)$.
b) Use the theorem on derivatives on inverses to find $(f^{-1})'(x)$ and compare it with the derivative that you get from the formula you find in a).

a) $y = x^2 + 4x - 7$

$x^2 + 4x - 7 - y = 0$

$$x = \frac{-4 \pm \sqrt{4^2 - 4\cdot 1\cdot(-7-y)}}{2\cdot 1}$$

$$= \frac{-4 \pm \sqrt{16 + 4(y+7)}}{2}$$

$$= \frac{-4 \pm \sqrt{4(4+y+7)}}{2} = \frac{-4 \pm 2\sqrt{11+y}}{2} = -2 \pm \sqrt{y+11}$$

Since $x \le -2$ we take the $-$ solution

$f^{-1}(y) = -2 - \sqrt{y+11}$,

b) $f^{-1}(x) = -2 - \sqrt{x+11}$

$(f^{-1})'(x) = -\frac{1}{2\sqrt{x+11}}$ from a)

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$ $[f'(x) = 2x+4]$

$$= \frac{1}{2(-2-\sqrt{x+11})+4}$$

$$= \frac{1}{-4-2\sqrt{x+11}+4} = -\frac{1}{2\sqrt{x+11}}$$

$^0$Total points: 100