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| **Calculus 1 — Exam 2** |
| **MAT 250, Fall 2019 — D. Ivanšić** |

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*Show all your work!*

Differentiate and simplify where appropriate:

**1.** (6pts) $\frac{d}{dx}\left(3x^7 - b^3 + \sqrt[5]{x^8} - \frac{7}{x^6}\right) = \frac{d}{dx}\left(3x^7 - b^3 + x^{\frac{8}{5}} - 7x^{-6}\right)$

$= 21x^6 - 0 + \frac{8}{5}x^{\frac{3}{5}} + 42x^{-7} = 21x^6 + \frac{8}{5}x^{\frac{3}{5}} + \frac{42}{x^7}$

**2.** (6pts) $\frac{d}{dx}\left(x\sqrt{x+3}\right) = 1 \cdot \sqrt{x+3} + x \cdot \frac{1}{2\sqrt{x+3}} = \frac{\sqrt{x+3} \cdot 2\sqrt{x+3} + x}{2\sqrt{x+3}} = \frac{2(x+3)+x}{2\sqrt{x+3}}$

$= \frac{3x+6}{2\sqrt{x+3}}$

**3.** (6pts) $\frac{d}{dt}\frac{t^2-1}{2t+5} = \frac{2t \cdot (2t+5) - (t^2-1) \cdot 2}{(2t+5)^2} = \frac{4t^2+10t-2t^2+2}{(2t+5)^2}$

$= \frac{2t^2+10t+2}{(2t+5)^2}$

**4.** (7pts) $\frac{d}{d\theta}\frac{\sin\theta}{\cos^3\theta} = \frac{\cos\theta \cdot \cos^3\theta - \sin\theta \cdot 3\cos^2\theta(-\sin\theta)}{\cos^6\theta} = \frac{\cos^4\theta + 3\sin^2\theta\cos^2\theta}{\cos^6\theta}$

$= \frac{\cancel{\cos^2\theta}(\cos^2\theta + 3\sin^2\theta)}{\cancel{\cos^6\theta}\ 4} = \frac{\cos^2\theta + 3\sin^2\theta}{\cos^4\theta}$

**5.** (6pts) $\frac{d}{dx}\sqrt[3]{\cos(x^2-7)} = \frac{d}{dx}\left(\cos(x^2-7)\right)^{\frac{1}{3}} = \frac{1}{3}\left(\cos(x^2-7)\right)^{-\frac{2}{3}}\left(-\sin(x^2-7)\right) \cdot 2x$

$= -\frac{2x\sin(x^2-7)}{3(\cos(x^2-7))^{\frac{2}{3}}}$

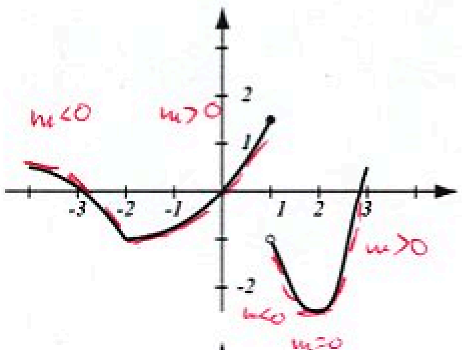
**6.** (6pts) The position function of an object is given by $s(t) = t^2 - \sin(2t)$. Write the velocity and acceleration functions for this motion.

$v(t) = s'(t) = 2t - 2\cos(2t)$

$a(t) = v'(t) = 2 + 4\sin(2t)$

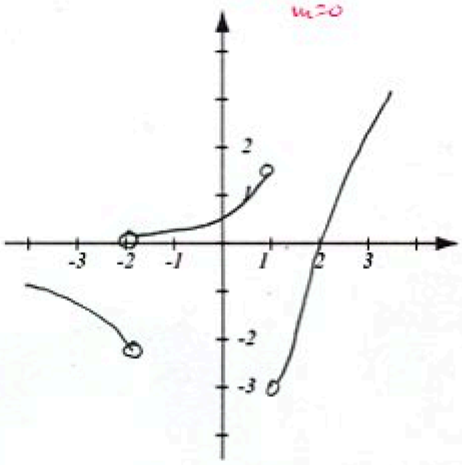
**7.** (10pts) The graph of the function $f(x)$ is shown at right.

a) Where is $f(x)$ not differentiable? Why?
b) Use the graph of $f(x)$ to draw an accurate graph of $f'(x)$.




a) At $x = -2$ sharp point
$x = 1$ discontinuous

b)




**8.** (13pts) Let $f(x) = \sqrt{x}$, and $x > 0$.
a) Use the limit definition of the derivative to find the derivative of the function.
b) Check your answer by taking the derivative of $f$ using differentiation rules.
c) Write the equation of the tangent line to the curve $y = f(x)$ at point $(9, 3)$.

a) $f'(a) = \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \cdot \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \lim_{x\to a} \frac{\sqrt{x}^2-\sqrt{a}^2}{(x-a)(\sqrt{x}+\sqrt{a})} = \lim_{x\to a} \frac{\cancel{x-a}}{(\cancel{x-a})(\sqrt{x}+\sqrt{a})}$

$= \frac{1}{\sqrt{a}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$

b) $f'(x) = \frac{1}{2\sqrt{x}}$ known fact

c) $f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

$y - 3 = \frac{1}{6}(x-9)$

$y = \frac{1}{6}x - \frac{9}{6} + 3$

$y = \frac{1}{6}x + \frac{3}{2}$

**9.** (10pts) Let $g(x) = \dfrac{f(x)}{x^2}$ and $h(x) = f(x \cdot f(x))$.
a) Find the general expressions for $g'(x)$ and $h'(x)$.
b) Use the table of values at right to find $g'(3)$ and $h'(2)$.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | -1 | 2 | 3 | -5 |
| $f'(x)$ | -2 | 3 | 4 | -1 |

a) $g'(x) = \dfrac{f'(x)\cdot x^2 - f(x)\cdot 2x}{x^4} = \dfrac{x f'(x) - 2f(x)}{x^3}$

$h'(x) = f'(xf(x))\cdot(x f(x))' = f'(xf(x))\,(1\cdot f(x) + x f'(x)) = f'(xf(x))\,(f(x) + x f'(x))$

b) $g'(3) = \dfrac{3\cdot 4 - 2\cdot 3}{3^3} = \dfrac{6}{3^3} = \dfrac{2}{9}$

$h'(2) = f'(2\cdot 2)(2 + 2\cdot 3)$
$= (-1)\cdot 8 = -8$

**10.** (8pts) Find the point on the curve $y = 2x^3 - 3x^2 - 31x + 7$ where the tangent line is parallel to the line $y = 5x - 17$.

$y' = 6x^2 - 6x - 31$

$y' = 5$

$6x^2 - 6x - 31 = 5$

$6x^2 - 6x - 36 = 0 \quad | \div 6$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3, -2$

**11.** (10pts) Use implicit differentiation to find $y'$.

$\sqrt{xy} = x^3 + y^3 - \tan y \quad \Big| \dfrac{d}{dx}$

$\dfrac{1}{2\sqrt{xy}}(1\cdot y + xy') = 3x^2 + 3y^2y' - \sec^2 y \cdot y' \quad | \cdot 2\sqrt{xy}$

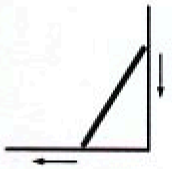
$y + xy' = 6x^2\sqrt{xy} + 6y^2\sqrt{xy}\,y' - 2\sqrt{xy}\sec^2 y \cdot y'$

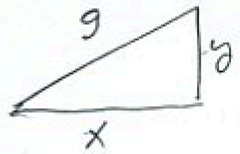
$xy' - 6y^2\sqrt{xy}\,y' + 2\sqrt{xy}\sec^2 y\, y' = 6x^2\sqrt{xy} - y$

$y'(\;\_\_\_\;) = (\qquad)$

$y' = \dfrac{6x^2\sqrt{xy} - y}{x - 6y^2\sqrt{xy} + 2\sqrt{xy}\sec^2 y}$

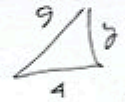
**12.** (12pts) A 9-foot ladder is sliding down the wall against which it is leaning. When the bottom of the ladder is 4 feet from the base of the wall, it is moving away from the wall at speed $\frac{1}{3}$ feet per second. How fast is the top of the ladder dropping at that moment?






Need: $y'$ when $x=4$,

Know $x' = \frac{1}{3}$




$y = \sqrt{9^2 - 4^2}$

$= \sqrt{81 - 16}$

$= \sqrt{65}$

$x^2 + y^2 = 9^2 \;\Big|\; \frac{d}{dt}$

$2xx' + 2yy' = 0$

$2yy' = -2xx'$

$y' = -\frac{xx'}{y}$

$y' = -\dfrac{4 \cdot \frac{1}{3}}{\sqrt{65}}$

$= -\dfrac{4}{3\sqrt{65}}$ ft/s

**Bonus.** (10pts) Find points on the circle $x^2 + y^2 = 20$ where the tangent line is parallel to the line $6x - 2y = 4$. Draw the circle, the given line and the parallel tangent line(s). *(Hint: implicit differentiation is a little easier.)*

$6x - 2y = 4$

$2y = 6x - 4 \;|\; \div 2$

$y = 3x - 2$

slope $= 3$

$x^2 + y^2 = 20 \;\Big|\; \frac{d}{dx}$

$2x + 2yy' = 0$

$y' = -\frac{x}{y}$

must equal 3

$\begin{cases} -\frac{x}{y} = 3 \Rightarrow x = -3y \\ x^2 + y^2 = 20 \end{cases}$

so $(-3y)^2 + y^2 = 20$

$10y^2 = 20$

$y^2 = 2$

$y = \pm\sqrt{2}$

Points are

$(-3\sqrt{2}, \sqrt{2})$

$(3\sqrt{2}, -\sqrt{2})$