Calculus 3 — Final Exam
MAT 309, Spring 2018 — D. Ivanšić

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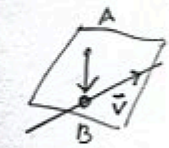
1. (12pts) Find the equation of the plane that contains the point $A = (3, 1, -2)$ and the line given by parametric equations $x = 3 - 2t$, $y = 4$, $z = -1 + t$.

Points in plane:



$A = (3,1,-2)$

$B = (3,4,-1)$

$\vec{AB} = \langle 0, 3, 1\rangle$

direction vector of line:

$\vec{v} = \langle -2, 0, 1\rangle$

$\vec{n} = \vec{v} \times \vec{AB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix} = -3\vec{i} - (-2)\vec{j} - 6\vec{k} =$

Take $\vec{n} = 3\vec{i} - 2\vec{j} + 6\vec{k}$

Equation of plane: $3(x-3) - 2(y-1) + 6(z+2) = 0$

$3x - 2y + 6z - 9 + 2 + 12 = 0$

$3x - 2y + 6z = -5$

2. (20pts) Let $f(x, y) = \dfrac{1}{x^2 + y^2}$.

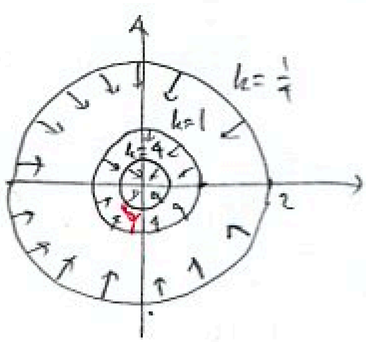
a) Sketch the contour map for the function, drawing level curves for levels $k = 0, \frac{1}{4}, 1, 4$.
b) At point $(1, -2)$, find the directional derivative of f in the direction of $\langle -2, -3\rangle$. In what direction is the directional derivative the greatest? What is the directional derivative in that direction?
c) Let $\mathbf{F} = \nabla f$. Sketch the vector field $\mathbf{F}$.
Apply the fundamental theorem for line integrals to answer:
d) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is the straight line from $(1, 1)$ to $(3, 0)$?
e) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is a curve going from any point on level curve $k = 4$ to any point on level curve $k = 1$?

a) $\dfrac{1}{x^2+y^2} = k$

$x^2 + y^2 = \dfrac{1}{k}$

circle with center $\dfrac{1}{\sqrt{k}}$, when $k > 0$

$k = 0$ nothing



b) $\nabla f = \left\langle -\dfrac{2x}{(x^2+y^2)^2}, -\dfrac{2y}{(x^2+y^2)^2} \right\rangle$

$D_{\vec{u}} f = \nabla f(1,-2) \cdot \dfrac{\vec{u}}{|\vec{u}|} = \left\langle -\dfrac{2}{25}, \dfrac{4}{25} \right\rangle \cdot \dfrac{\langle -2,-3\rangle}{\sqrt{(-2)^2+(-3)^2}}$

$= \frac{1}{25}\langle -2, 4\rangle \cdot \frac{1}{\sqrt{13}}\langle -2, -3\rangle = \dfrac{1}{25\sqrt{13}}(4-12) = -\dfrac{8}{25\sqrt{13}}$

Greatest in direction $\nabla f = \frac{1}{25}\langle -2, 4\rangle$

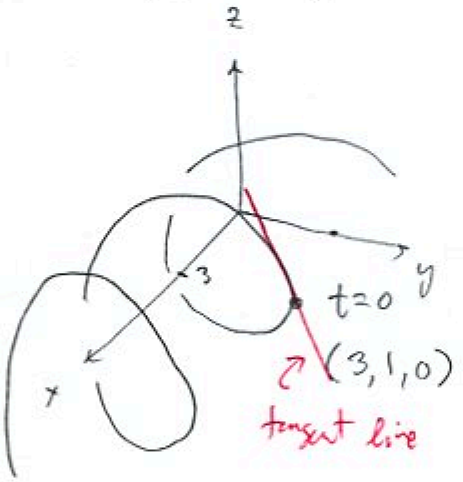
with value $|\nabla f| = \frac{1}{25}\sqrt{(-2)^2 + 4^2} = \dfrac{\sqrt{20}}{25} = \dfrac{2\sqrt{5}}{25}$

d) $\int_C \nabla f \cdot d\vec{r} = f(3,0) - f(1,1) = \dfrac{1}{9} - \dfrac{1}{2} = -\dfrac{7}{18}$

e) $\int_C \nabla f \cdot d\vec{r} = 1 - 4 = -3$

3. (14pts) The curve $\mathbf{r}(t) = \langle t+3, \cos t, \sin t\rangle$ is given, t any real number.
a) Sketch the curve in the coordinate system.
b) Find parametric equations of the tangent line to this curve when $t = 0$ and sketch the tangent line.

a) Rotates in yz-plane
Moves linearly in x-direction



$\vec{r}(0) = \langle 3,1,0\rangle$

$\vec{r}\,'(t) = \langle 1, -\sin t, \cos t\rangle$

$\vec{r}\,'(0) = \langle 1, 0, 1\rangle$

$x = 3+t$
$y = 1$
$z = t$

4. (14pts) Find and classify the local extremes for $f(x, y) = 2x^2 - 5y^2 - 2xy + 6x - 14y$.

$\nabla f = \langle 4x - 2y + 6, -10y - 2x - 14\rangle$

$\nabla f = \vec{0}$

$$\begin{cases} 4x - 2y + 6 = 0 \\ -2x - 10y - 14 = 0 \end{cases}$$

$2x - y = -3$
$x + 5y = -7$
$-11y = 11$ $\quad 1 + (-2) \cdot 11$
$y = -1$
$x = -7 - 5(-1) = -2$

Candidates: $(-2,-1)$

$D = \begin{vmatrix} 4 & -2 \\ -2 & -10 \end{vmatrix} = -40 - 4 = -44$

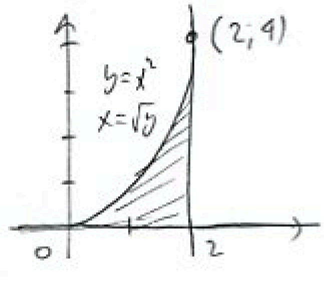
$D < 0$ always so we have a saddle point at $(-2,-1)$

5. (16pts) Let D be the region in the first quadrant bounded by the curves $y = x^2$, $y = 0$ and $x = 2$.

a) Sketch the region D.

b) Set up $\iint_D e^{x^3}\, dA$ as iterated integrals in both orders of integration.

c) Evaluate the double integral using the easier order.



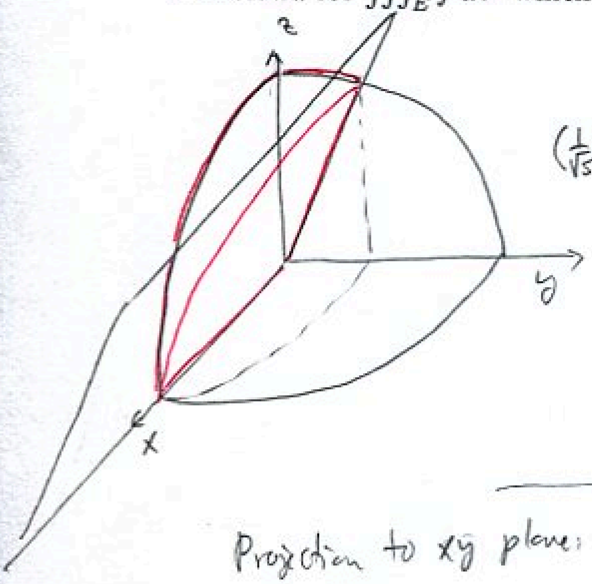
Type 1: $\int_0^2\int_0^{x^2} e^{x^3}\, dy\, dx$ Type 2: $\int_0^4\int_{\sqrt{y}}^{4} e^{x^3}\, dx\, dy$

↓ easier:

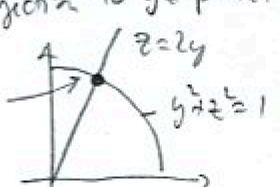
$$\int_0^2 e^{x^3}(x^2-0)\, dx = \int_0^2 x^2 e^{x^3}\, dx = \frac{e^{x^3}}{3}\Big|_0^2$$

$$= \frac{1}{3}(e^8-1) = \frac{e^8-1}{3}$$

6. (18pts) Sketch the region E in the first octant $(x, y, z \geq 0)$ that is inside the sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 2y$. Then write the two iterated triple integrals that stand for $\iiint_E f\, dV$ which end in $dx\, dz\, dy$ and $dz\, dy\, dx$.



Projection to yz plane:



$$\int_0^{\frac{1}{\sqrt5}}\int_{2y}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-y^2-z^2}} f\, dx\, dz\, dy$$

$y^2+(2y)^2=1$

$5y^2=1$

$y = \pm\frac{1}{\sqrt5}$

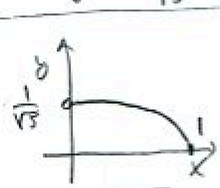
Projection to xy plane:

$\begin{cases} z=2y \\ x^2+y^2+z^2=1 \end{cases}$

$x^2+y^2+(2y)^2=1$

$x^2+5y^2=1$

ellipse

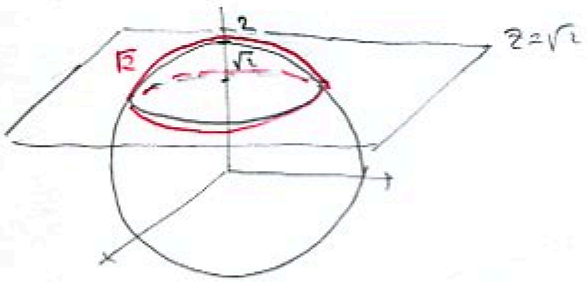


$\frac{x^2}{1}+\frac{y^2}{\frac15}=1$

$y=\sqrt{\frac{1-x^2}{5}}$

$$\int_0^1\int_0^{\sqrt{\frac{1-x^2}{5}}}\int_{2y}^{\sqrt{1-x^2-y^2}} f\, dz\, dy\, dx$$

7. (14pts) Use either cylindrical or spherical coordinates to set up $\displaystyle\iiint_E \frac{x^2+y^2+z^2}{x^2+1}\,dV$, where E is the region inside the sphere $x^2+y^2+z^2=4$ and above the plane $z=\sqrt{2}$. Simplify the expression but do NOT evaluate the integral. Sketch the region E.



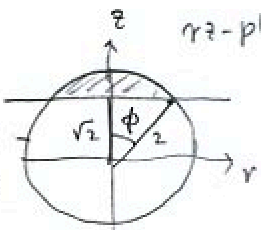
Cylindrical:

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{\sqrt{2}}^{\sqrt{4-r^2}} \frac{r^2+z^2}{r^2\cos^2\theta+1}\cdot r\,dz\,dr\,d\theta$$

rz-plane

$r^2+z^2=4$

$z=\pm\sqrt{4-r^2}$



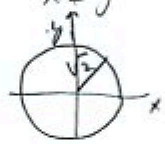
$\cos\phi=\frac{\sqrt{2}}{2}$, $\phi=\frac{\pi}{4}$

Proj. to xy-plane:

$\begin{cases} x^2+y^2+z^2=4 \\ z=\sqrt{2} \end{cases}$

$x^2+y^2+2=4$

$x^2+y^2=2$



Spherical,

$$\int_0^{2\pi}\int_0^{\pi/4}\int_{\frac{\sqrt{2}}{\cos\phi}}^{2} \frac{\rho^4\sin\phi}{\rho^2\sin^2\phi\cos^2\theta+1}\,d\rho\,d\phi\,d\theta$$

$z=\sqrt{2}$

$\rho\cos\phi=\sqrt{2}$

$\rho=\frac{\sqrt{2}}{\cos\phi}$

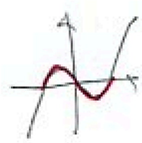
$$\frac{\rho^2}{(\rho\sin\phi\cos\theta)^2+1}\cdot\rho^2\sin\phi$$

8. (10pts) Set up and simplify the set-up, but do NOT evaluate the integral: $\displaystyle\int_C \frac{x+y}{xy+1}\,ds$, where C is the part of the curve $y=x^3-x$ from $(-1,0)$ to $(1,0)$.

Parametrization:

$x=t$ $\quad -1\le t\le 1$

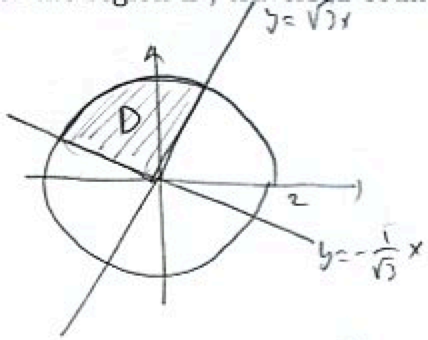
$y=t^3-t$



$\vec{r}\,'(t)=\langle 1, 3t^2-1\rangle$

$$\int_C \frac{x+y}{xy+1}\,ds=\int_{-1}^{1}\frac{t+t^3-t}{t(t^3-t)+1}\sqrt{1^2+(3t^2-1)^2}\,dt=\int_{-1}^{1}\frac{t^3}{t^4-t^2+1}\sqrt{9t^4-6t^2+2}$$

9. (20pts) Consider the region inside the circle $x^2 + y^2 = 4$ and above the lines $y = \sqrt{3}x$ and $y = -\frac{1}{\sqrt{3}}x$.
a) Draw the region.
b) Use Green's theorem to find the line integral $\int_C y^3\,dx + x^3\,dy$, where C is the boundary of the region D, traversed counterclockwise.



$$\int_C y^3 dx + x^3 dy = \iint_D \frac{\partial}{\partial x} x^3 - \frac{\partial}{\partial y} y^3 \, dA$$

$$= \iint_D 3x^2 - 3y^2 \, dA = \left[\begin{matrix}\text{switch to}\\ \text{polar}\end{matrix}\right]$$

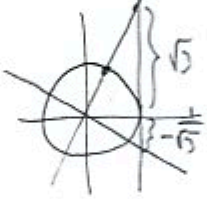
$\tan\theta = \sqrt{3}$, $\theta = \frac{\pi}{3}$

$\tan\theta = -\frac{1}{\sqrt{3}}$, $\theta = -\frac{\pi}{6}, \frac{5\pi}{6}$

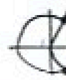
$$= 3\int_{\pi/3}^{5\pi/6}\int_0^2 \left(r^2\cos^2\theta - r^2\sin^2\theta\right) r\,dr\,d\theta$$

$\cos^2\theta - \sin^2\theta = \cos(2\theta)$

$$= 3\int_{\pi/3}^{5\pi/6}\int_0^2 r^3\cos(2\theta)\,dr\,d\theta = 3\int_0^2 r^3 dr \int_{\pi/3}^{5\pi/6}\cos(2\theta)\,d\theta$$



$$= 3\cdot\frac{r^4}{4}\Big|_0^2 \cdot \frac{\sin(2\theta)}{2}\Big|_{\pi/3}^{5\pi/6} = 3\cdot\frac{16}{4}\cdot\frac{1}{2}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = -6\sqrt{3}$$

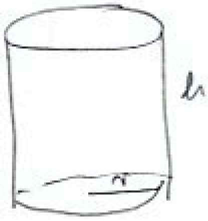


$\sin\frac{5\pi}{3} - \sin\frac{\pi}{3}$

10. (12pts) Use differentials to estimate the change in the volume of a cylindrical can if its height decreases from 28 to 26 centimeters, and its radius increases from 14 to 15 centimeters.

$V = \pi r^2 h$

$$\Delta V \approx dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$



$$= 2\pi r h\,dr + \pi r^2\,dh$$

$$= 2\pi\cdot 14\cdot 28\cdot \underset{15-14}{1} + \pi\cdot 14^2\cdot \underset{26-28}{(-2)}$$

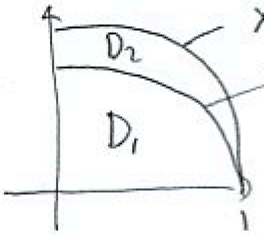
$$= \pi\left(28^2 - 2\cdot 14^2\right) = \pi\cdot 28(28-14) = 28\cdot 14\cdot\pi$$

$$= 392\pi$$

$$\begin{array}{r} 28\cdot 14 \\ \hline 28 \\ 112 \\ \hline 392 \end{array}$$

Bonus (10pts) Do problem 4 for the iterated triple integral that ends in $dy\,dz\,dx$.

Projection to xz plane



$x^2+z^2=1$

$x^2+\frac{5}{4}z^2=1$

$z=\sqrt{\frac{4}{5}(1-x^2)}$

$$\iiint_E f\,dV = \iint_{D_1}\int_0^{\frac{z}{2}} f\,dy\,dA + \iint_{D_2}\int_0^{\sqrt{1-x^2-z^2}} f\,dy\,dA$$

$$= \int_0^1\int_0^{\sqrt{\frac{4}{5}(1-x^2)}}\int_0^{\frac{z}{2}} f\,dy\,dz\,dx$$

$$+ \int_0^1\int_{\sqrt{\frac{4}{5}(1-x^2)}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-z^2}} f\,dy\,dz\,dx$$

$x^2+y^2+z^2=1$

$z=2y \Leftrightarrow y=\frac{z}{2}$

$x^2+\left(\frac{z}{2}\right)^2+z^2=1$

$x^2+\frac{5}{4}z^2=1$