Calculus 3 — Exam 4
MAT 309, Spring 2018 — D. Ivanšić

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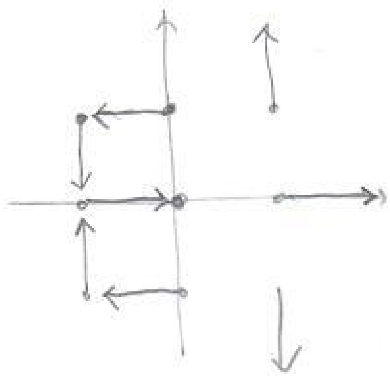
Show all your work!

1. (12pts) Let $\mathbf{F}(x,y) = \langle x^2 - y^2, xy \rangle$.
a) Sketch the vector field by evaluating it at 9 points (for example, a 3×3 grid).
b) Is F conservative?

a)

(x,y)	$F(x,y)$
$(-1,-1)$	$\langle 0,1\rangle$
$(0,-1)$	$\langle -1,0\rangle$
$(1,-1)$	$\langle 0,-1\rangle$
$(-1,0)$	$\langle 1,0\rangle$
$(0,0)$	$\langle 0,0\rangle$
$(1,0)$	$\langle 1,0\rangle$
$(-1,1)$	$\langle 0,-1\rangle$
$(0,1)$	$\langle -1,0\rangle$
$(1,1)$	$\langle 0,1\rangle$



b) $\frac{\partial Q}{\partial x} \stackrel{?}{=} \frac{\partial P}{\partial y}$

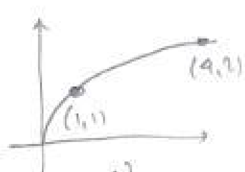
$\frac{\partial}{\partial x} xy \stackrel{?}{=} \frac{\partial}{\partial y}(x^2 - y^2)$

$y \neq -2y$

So not conservative

2. (20pts) In both cases set up and simplify the set-up, but do NOT evaluate the integral.
a) $\int_C \frac{xy}{x^2+y^2}\, ds$, where C is the part of the parabola $y = \sqrt{x}$ from $(1,1)$ to $(4,2)$.
b) $\int_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F}(x,y,z) = \left\langle -z, -z, \frac{x-y}{x^2+y^2+z^2} \right\rangle$, where C is the curve
$x = \cos t,\ y = -\cos t,\ z = \sqrt{2}\sin t,\ 0 \le t \le 2\pi$.

a)



$x = t^2$ $\quad 1 \le t \le 2$
$y = t$
$x' = 2t$
$y' = 1$

$\int_C \frac{xy}{x^2+y^2}\, ds = \int_1^2 \frac{t^2 \cdot t}{(t^2)^2 + t^2} \cdot \sqrt{(2t)^2 + 1^2}\, dt = \int_1^2 \frac{t^3}{t^4+t^2}\sqrt{4t^2+1}\, dt$

$= \int_1^2 \frac{t}{t^2+1}\sqrt{4t^2+1}\, dt$

b) $\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left\langle -\sqrt{2}\sin t, -\sqrt{2}\sin t, \frac{\cos t - (-\cos t)}{\cos^2 t + (-\cos t)^2 + (\sqrt{2}\sin t)^2} \right\rangle \cdot \langle -\sin t, \sin t, \sqrt{2}\cos t \rangle\, dt$

$= \int_0^{2\pi} \sqrt{2}\sin^2 t - \sqrt{2}\sin^2 t + \frac{2\cos t \cdot \sqrt{2}\cos t}{\underbrace{2\cos^2 t + 2\sin^2 t}_{=2}}\, dt$

$= \int_0^{2\pi} \sqrt{2}\cos^2 t\, dt$

3. (12pts) Let $\mathbf{F}(x,y) = \langle 6x+7y, 7x-10y\rangle$. It is easy to see that $\mathbf{F} = \nabla f$, where $f(x,y) = 3x^2+7xy-5y^2$. Apply the fundamental theorem for line integrals to:
a) Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, if C is the unit circle.
b) Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, if C is a curve from $(0,0)$ to $(3,1)$. (Why is the curve not specified?)

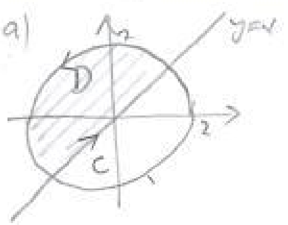
a) $\int_C \vec{F}\cdot d\vec{r} = 0$ because line integral is independent of curve

b) For same reason $\int_C \vec{F}\cdot d\vec{r}$ does not depend on which curve C is.

$$\int_C \vec{F}\cdot d\vec{r} = f(3,1) - f(0,0) = 3\cdot 3^2 + 7\cdot 3\cdot 1 - 5\cdot 1^2 - 0 = 27+21-5 = 43$$

4. (20pts) Consider the region inside the circle $x^2+y^2=4$ and above the line $y=x$.
a) Draw the region.
b) Use Green's theorem to find the line integral $\int_C y^2\,dx + \frac{x^2}{2}\,dy$, where C is the boundary of the region D, traversed counterclockwise.

a)



b)
$$\int_C y^2\,dx + \frac{x^2}{2}\,dy = \iint_D \left(\frac{\partial}{\partial x}\frac{x^2}{2} - \frac{\partial}{\partial y}y^2\right)dA$$

$$= \iint_D x - 2y\,dA = \begin{bmatrix}\text{switch to}\\ \text{polar}\end{bmatrix}$$

$$= \int_{\pi/4}^{5\pi/4}\int_0^2 (r\cos\theta - 2r\sin\theta)\,r\,dr\,d\theta =$$

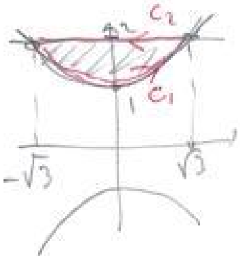
$$= \int_{\pi/4}^{5\pi/4}\int_0^2 (\cos\theta - 2\sin\theta)\cdot r^2\,dr\,d\theta$$

$$= \int_{\pi/4}^{5\pi/4} (\cos\theta - 2\sin\theta)\cdot \frac{r^3}{3}\Big|_0^2 d\theta = \frac{8}{3}\int_{\pi/4}^{5\pi/4} \cos\theta - 2\sin\theta\,d\theta$$

$$= \frac{8}{3}\left(\sin\theta + 2\cos\theta\Big|_{\pi/4}^{5\pi/4}\right) = \frac{8}{3}\left(\underbrace{-\frac{\sqrt2}{2} - \frac{\sqrt2}{2}}_{=-\sqrt2} + 2\underbrace{\left(-\frac{\sqrt2}{2} - \frac{\sqrt2}{2}\right)}_{-\sqrt2}\right) = \frac{8}{3}\cdot(-3\sqrt2)$$

$$= \boxed{-8\sqrt2}$$

5. (22pts) Let D be the region enclosed by the hyperbola $y^2 - x^2 = 1$ and the line $y = 2$. Draw the region.
a) Write the double integral for the area of D.
b) Use Green's theorem to write the integrals that give the area of D.
In both a) and b), simplify until you encounter a hard integral.



$y^2 - x^2 = 1$
$2^2 - x^2 = 1$
$x^2 = 3$
$x = \pm\sqrt{3}$

a) $\text{Area} = \iint_D 1\, dA = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{\sqrt{x^2+1}}^{2} 1\, dy\, dx = \int_{-\sqrt{3}}^{\sqrt{3}} 2 - \sqrt{x^2+1}\, dx$

$= 4\sqrt{3} - \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{x^2+1}\, dx$

$y^2 = x^2 + 1$
$y = \sqrt{x^2+1}$

b) $\text{Area} = \int_{C_1 \cup C_2} x\, dy = \int_{C_1} x\, dy - \int_{-C_2} x\, dy$

$C_1: \; x = t, \; y = \sqrt{t^2+1} \quad -\sqrt{3} \le t \le \sqrt{3}$

$-C_2: \; x = t, \; y = 2 \quad -\sqrt{3} \le t \le \sqrt{3}$

$= \int_{-\sqrt{3}}^{\sqrt{3}} t \cdot \frac{2t}{\sqrt{t^2+1}}\, dt - \int_{-\sqrt{3}}^{\sqrt{3}} t \cdot 0\, dt$

$= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{t^2}{\sqrt{t^2+1}}\, dt$

6. (14pts) Let $\mathbf{F}(x, y, z) = \langle z^2 - y, \cos z - x, 2xz - y \sin z \rangle$.
a) Find the curl of $\mathbf{F}$.
b) Is $\mathbf{F}$ is conservative? If so, find its potential function.

$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - y & \cos z - x & 2xz - y\sin z \end{vmatrix} = (-\sin z - (-\sin z))\vec{i} - (2z - 2z)\vec{j} + (-1-(-1))\vec{k}$

$= \vec{0}$

Since the domain is $\mathbb{R}^3$ (simply connected), curl $\vec{F} = \vec{0}$ says it is conservative

$\vec{F} = \nabla f$, $\quad \frac{\partial f}{\partial x} = z^2 - y$

$f = z^2 x - yx + g(y, z)$

$\cos z - x = \frac{\partial f}{\partial y} = -x + \frac{\partial g}{\partial y}$

$\Rightarrow \frac{\partial g}{\partial y} = \cos z$

$g(y, z) = y \cos z + h(z)$

$f = z^2 x - yx + y\cos z + h(z)$

$2xz - y\sin z = \frac{\partial f}{\partial z} = 2zx - y\sin z + h'(z)$

$\Rightarrow h'(z) = 0$, so $h(z) = C$

$\boxed{f(x, y, z) = z^2 x - yx + y\cos z + C}$

Bonus. (10pts) Show that the expressions you got in a) and b) of problem 5 are the same number. (You don't have to evaluate the integrals to see this — manipulate one to get the other.)

We need to show: $4\sqrt{3} - \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{x^2+1}\, dx = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{t^2}{\sqrt{t^2+1}}\, dt$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{t^2}{\sqrt{t^2+1}}\, dt = \begin{bmatrix} u = t & dv = \frac{t}{\sqrt{t^2+1}}\, dt \\ du = 1\, dt & v = \sqrt{t^2+1} \end{bmatrix} = t\sqrt{t^2+1}\Big|_{-\sqrt{3}}^{\sqrt{3}} - \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{t^2+1}\, dt$$

$$= \sqrt{3}\cdot\sqrt{4} - (-\sqrt{3})\sqrt{4} - \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{t^2+1}\, dt$$

$$= 2\sqrt{3} + 2\sqrt{3} - \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{t^2+1}\, dt$$

$$= 4\sqrt{3} - \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{x^2+1}\, dx$$