Calculus 3 — Exam 3
MAT 309, Spring 2018 — D. Ivanšić

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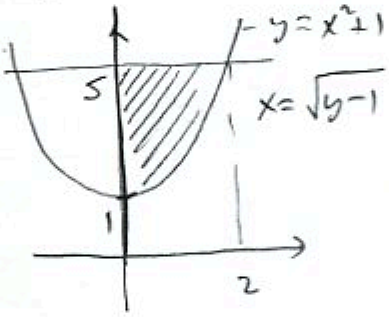
Show all your work!

1. (17pts) Let D be the region bounded by the curves $y = x^2 + 1$, $x = 0$ and $y = 5$.
a) Sketch the region D.
b) Set up $\iint_D \frac{1}{\sqrt{y-1}}\, dA$ as iterated integrals in both orders of integration.
c) Evaluate the double integral using the easier order.



Type 1: $\int_0^2 \int_{x^2+1}^5 \frac{1}{\sqrt{y-1}}\, dy\, dx$ Type 2: $\int_1^5 \int_0^{\sqrt{y-1}} \frac{1}{\sqrt{y-1}}\, dx\, dy$

const. for x

Type 2 easier: $\int_1^5 \frac{1}{\sqrt{y-1}}(\sqrt{y-1}-0)\, dy$

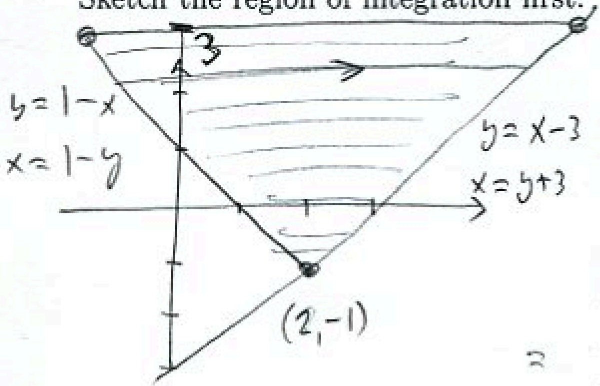
$= \int_1^5 dy = 1\cdot(5-1) = 4$

$x^2+1=5$

$x^2=4$

$x=\pm 2$

2. (17pts) Find $\iint_D xy\, dA$ if D is the triangle bounded by $y = 1 - x$, $y = x + 3$ and $y = 3$. Sketch the region of integration first.



As type 2 region:

$\int_{-1}^3 \int_{1-y}^{y+3} xy\, dx\, dy = \int_{-1}^3 y \left.\frac{x^2}{2}\right|_{x=1-y}^{y+3} dy$

$= \int_{-1}^3 \frac{y}{2}\left((y+3)^2 - (1-y)^2\right) dy$

$= \int_{-1}^3 \frac{y}{2}\left(y^2+6y+9-(1-2y+y^2)\right) dy$

$= \int_{-1}^3 \frac{y}{2}(8y+8)\, dy = \int_{-1}^3 4y^2+4y\, dy$

$= 4\frac{y^3}{3} + 4\frac{y^2}{2}\Big|_{-1}^3 = \frac{4}{3}(3^3-(-1)^3) + 2(3^2-(-1)^2)$

$= \frac{4}{3}\cdot 28 + 16 = \frac{112+48}{3} = \frac{160}{3}$

$y=1-x$

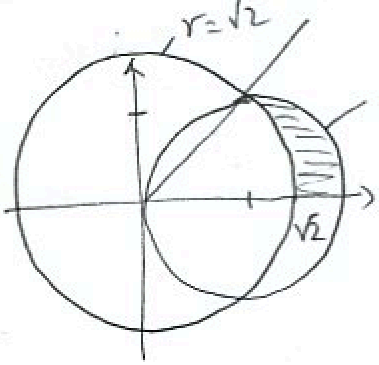
$y=x-3$

$2y=-2$

$y=-1$

$x=2$

3. (20pts) Use polar coordinates to find $\iint_D \frac{y}{\sqrt{x^2+y^2}}\,dA$, if D is the region inside the circle $(x-1)^2+y^2=1$, outside the circle $\underbrace{x^2+y^2=2}_{\text{radius}=\sqrt{2}}$ and above the x-axis. Sketch the region of integration first.



$$\int_0^{\frac{\pi}{4}}\int_{\sqrt{2}}^{2\cos\theta} \frac{r\sin\theta}{r}\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_{\sqrt{2}}^{2\cos\theta} r\sin\theta\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \sin\theta \left.\frac{r^2}{2}\right|_{\sqrt{2}}^{2\cos\theta} d\theta$$

$2\cos\theta = \sqrt{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\theta = \frac{\pi}{4}$

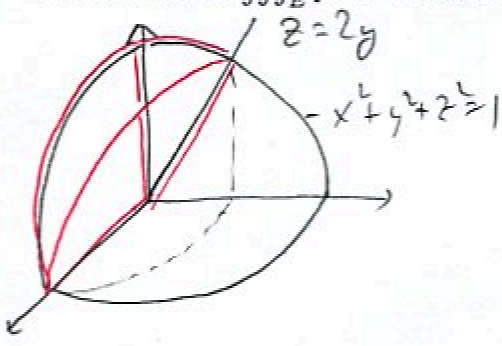
$$= \int_0^{\pi/4} \frac{\sin\theta}{2}\left(4\cos^2\theta - 2\right) d\theta = \begin{bmatrix} u=\cos\theta & \theta=\frac{\pi}{4}, u=\frac{\sqrt{2}}{2} \\ du=-\sin\theta\,d\theta & \theta=0, u=1 \end{bmatrix}$$

$$= \int_1^{\sqrt{2}/2} -(2u^2-1)\,du = \int_{\sqrt{2}/2}^{1} 2u^2-1\,du = \left(\frac{2}{3}u^3-u\right)\Big|_{\sqrt{2}/2}^{1}$$

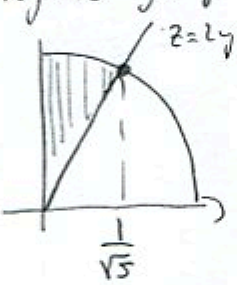
$$= \frac{2}{3}\left(1-\frac{2\sqrt{2}}{8}\right)-\left(1-\frac{\sqrt{2}}{2}\right) = -\frac{1}{3}-\frac{\sqrt{2}}{6}+\frac{\sqrt{2}}{2} = \frac{-\sqrt{2}+3\sqrt{2}}{6}-\frac{1}{3}$$

$$= \frac{\sqrt{2}}{3}-\frac{1}{3} = \frac{\sqrt{2}-1}{3}$$

4. (18pts) Sketch the region E in the first octant $(x,y,z \geq 0)$ that is inside the sphere $x^2+y^2+z^2=1$ and above the plane $z=2y$. Then write the two iterated triple integrals that stand for $\iiint_E f\,dV$ which end in $dx\,dz\,dy$ and $dz\,dy\,dx$.



Proj. to yz plane: $(x=0)$



$0^2+y^2+z^2=1$

$z=2y$

$y^2+4y^2=1$

$y=\pm\frac{1}{\sqrt{5}}$

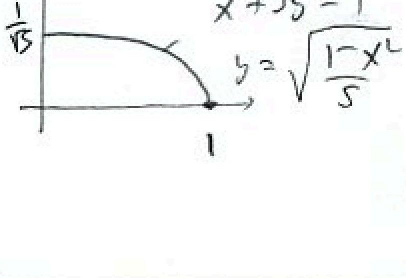
$$\int_0^{\frac{1}{\sqrt{5}}}\int_{2y}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-y^2-z^2}} f\,dx\,dz\,dy$$

Proj. to xy-plane

$$\begin{cases} z=2y \\ x^2+y^2+z^2=1 \end{cases}$$

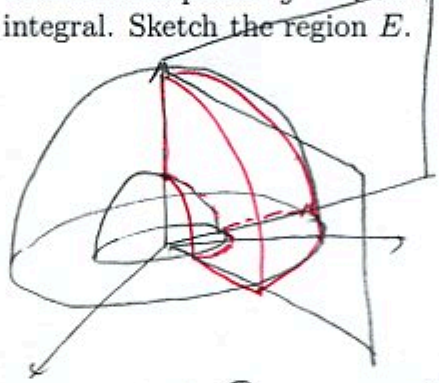
$x^2+y^2+(2y)^2=1$

$x^2+5y^2=1$



$$\int_0^1\int_0^{\sqrt{\frac{1-x^2}{5}}}\int_{2y}^{\sqrt{1-x^2-y^2}} f\,dz\,dy\,dx$$

5. (14pts) Use spherical coordinates to set up the triple integral for the volume of the region that is between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=25$, above the xy-plane, and between the planes $y=\sqrt{3}x$ and $y=-\sqrt{3}x$, the part where $y \geq 0$. Do not evaluate the integral. Sketch the region E.



$$\text{Vol } E = \iiint_E 1 \, dV$$

$$= \int_{\pi/3}^{2\pi/3} \int_0^{\pi/2} \int_2^5 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$0 \leq \phi \leq \frac{\pi}{2}$

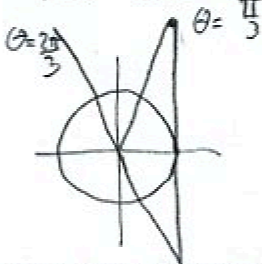
$2 \leq \rho \leq 5$

$\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}$

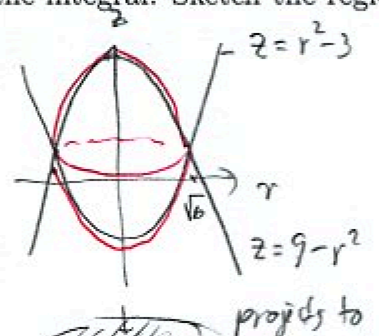
$y = \pm\sqrt{3}x$

$\frac{y}{x} = \pm\sqrt{3}$

$\tan\theta = \pm\sqrt{3}$

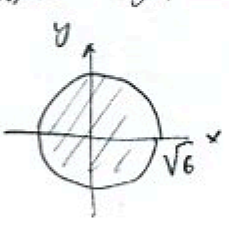


6. (14pts) Use cylindrical coordinates to set up $\iiint_E \frac{x^2+y^2+z^2}{x^2+y^2+1}\, dV$, where E is the region bounded by the paraboloids $z=x^2+y^2-3$ and $z=9-x^2-y^2$. Do not evaluate the integral. Sketch the region E.



$$\iiint_E f \, dV = \iint_D \int_{r^2-3}^{9-r^2} f \, dV$$

$$= \int_0^{2\pi} \int_0^{\sqrt{6}} \int_{r^2-3}^{9-r^2} \frac{r^2+z^2}{r^2+1} \, r \, dz \, dr \, d\theta$$



$\begin{cases} z = r^2-3 \\ z = 9-r^2 \end{cases}$

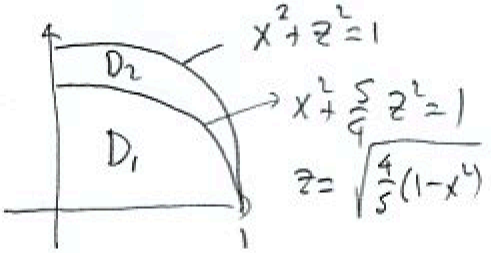
$9-r^2 = r^2-3$

$2r^2 = 12$

$r^2 = 6, \quad r = \pm\sqrt{6}$

Bonus (10pts) Do problem 4 for the iterated triple integral that ends in $dy\,dz\,dx$.

Projection to xz plane



$x^2+y^2+z^2=1$

$z=2y \Leftrightarrow y=\frac{z}{2}$

$x^2+\left(\frac{z}{2}\right)^2+z^2=1$

$x^2+\frac{5}{4}z^2=1$

$$\iiint_E f\,dV = \iint_{D_1}\int_0^{\frac{z}{2}} f\,dy\,dA + \iint_{D_2}\int_0^{\sqrt{1-x^2-z^2}} f\,dy\,dA$$

$$= \int_0^1\int_0^{\sqrt{\frac{4}{5}(1-x^2)}}\int_0^{\frac{z}{2}} f\,dy\,dz\,dx$$

$$+ \int_0^1\int_{\sqrt{\frac{4}{5}(1-x^2)}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-z^2}} f\,dy\,dz\,dx$$