Calculus 3 — Exam 2	
MAT 309, Spring 2018 — D. Ivanšić	

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1. (10pts) Let $h(x,y) = x^2 + 4y^2$.
a) Find the domain of h.
b) Sketch the contour map for the function, drawing level curves for levels $k = -1, 0, 1, 4$. Note the domain on the picture.
c) Without computation, draw the directions of $\nabla h(1,0)$ and $\nabla h(\sqrt{2}, \frac{1}{\sqrt{2}})$. Note that these points are on the level curves you drew in b).

a) domain $= \mathbb{R}^2$

b) $\underbrace{x^2+4y^2}_{\geq 0} = k$

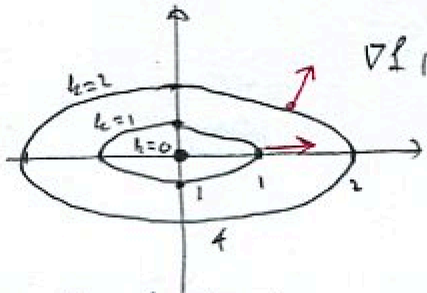
nothing for $k<0$

$(0,0)$ for $k=0$

ellipse for $k>0$

$\frac{x^2}{k} + \frac{y^2}{\frac{k}{4}} = 1$

semiaxes $\sqrt{k}$, $\frac{\sqrt{k}}{2}$



nothing for $k=-1$

∇f is perpendicular to level curves

2. (16pts) Let $f(x,y) = \dfrac{\sin^2 x}{\cos^2 y}$. $= \sin^2 x \cos^{-2} y$
a) At point $(\frac{\pi}{4}, \frac{\pi}{3})$, find the directional derivative of f in the direction of $\langle 1, 3\rangle$.
b) In what direction is the directional derivative the greatest, and what is its value?

a) $\nabla f = \left\langle \frac{1}{\cos^2 y} \cdot 2\sin x \cos x, \ \sin^2 x \cdot (-2\cos^{-3} y)(-\sin y) \right\rangle$

$= \left\langle \frac{2\sin x \cos x}{\cos^2 y}, \ \frac{2\sin^2 x \sin y}{\cos^3 y} \right\rangle$

$\nabla f(\frac{\pi}{4}, \frac{\pi}{3}) = \left\langle \frac{\cancel{2} \cdot \frac{\sqrt{2}}{\cancel{2}} \cdot \frac{\sqrt{2}}{2}}{(\frac{1}{2})^2}, \ \frac{2 (\frac{\sqrt{2}}{2})^2 \frac{\sqrt{3}}{2}}{(\frac{1}{2})^3} \right\rangle = \left\langle 4, \frac{2 \cdot \frac{2}{4} \cdot \frac{\sqrt{3}}{2}}{\frac{1}{8}} \right\rangle = \langle 4, 4\sqrt{3} \rangle$

$\vec{u} = \frac{1}{\sqrt{3^2+1^2}} \langle 1, 3 \rangle$ $\qquad \nabla f(\frac{\pi}{4}, \frac{\pi}{3}) \cdot \frac{1}{\sqrt{10}} \cdot \langle 1, 3\rangle = \frac{1}{\sqrt{10}}(1 \cdot 4 + 3 \cdot 4\sqrt{3}) = \frac{4+12\sqrt{3}}{\sqrt{10}}$

$= \frac{1}{\sqrt{10}} \langle 1, 3 \rangle$

b) in direction of $\nabla f = \langle 4, 4\sqrt{3} \rangle$

value is $|\nabla f| = \sqrt{16 + 16 \cdot 3} = \sqrt{64} = 8$

3. (12pts) Find the equation of the tangent plane to the hyperbolic paraboloid $y = 2x^2 - 3z^2$ at the point $(-1, -10, 2)$. Simplify the equation to standard form.

$2x^2 - y - 3z^2 = 0$

$F(x,y,z) = 2x^2 - y - 3z^2$

$\nabla F(x,y,z) = \langle 4x, -1, -6z \rangle$

normal vector is $\nabla F(-1,-10,2\rangle = \langle -4, -1, -12 \rangle$, use $\vec{n} = \langle 4, 1, 12 \rangle$

$4(x-(-1)) + (y-(-10)) + 12(z-2) = 0$

$4x + y + 12z = 10$

4. (18pts) Let $W = \dfrac{x}{y-x}$, $x = te^{st}$, $y = s^2 + t^2$. Use the chain rule to find $\dfrac{\partial W}{\partial t}$ when $s = 0$, $t = 2$.

When $s=0, t=2$

$x = 2, y = 4$

$\dfrac{\partial w}{\partial x} = \dfrac{1 \cdot (y-x) - x \cdot (-1)}{(y-x)^2} = \dfrac{y}{(y-x)^2}$

$\dfrac{\partial w}{\partial y} = x \cdot -(y-x)^{-2} = -\dfrac{x}{(y-x)^2}$

$\dfrac{\partial x}{\partial t} = e^{st} + te^{st} \cdot s = e^{ts}(1+ts)$

$\dfrac{\partial y}{\partial t} = 2t$

$\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x} \cdot \dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y} \cdot \dfrac{\partial y}{\partial t}$

$= \dfrac{4}{(4-2)^2} \cdot \underbrace{e^{0 \cdot 2}(1+0 \cdot 2)}_{=1} + \left(-\dfrac{2}{(4-2)}\right) \cdot 2 \cdot 2$

$= 1 - 2 = -1$

5. (12pts) The body mass index or BMI is calculated using the formula $BMI = \frac{w}{h^2}$, where w and h are weight and height of an individual in kilograms and meters, respectively. Use differentials to estimate the change in BMI if a 1-meter high child weighing 15 kg grows by 1.5cm in height and 0.5kg in weight.

$$dB = \frac{\partial B}{\partial w}\, dw + \frac{\partial B}{\partial h} \cdot dh$$

$h = 1 \quad dh = 0{,}015$

$w = 15 \quad dw = 0.5$

$$= \frac{1}{h^2}\, dw + w \cdot (-2h^{-3})\, dh$$

$$= \frac{1}{h^2}\, dw - \frac{2w}{h^3}\, dh$$

evaluated $= \frac{1}{1^2} \cdot 0{,}5 - \frac{2 \cdot 15}{1^3} \cdot 0{,}015 = 0{,}5 - 30 \cdot 0{,}015 = 0{,}5 - 0{,}455 = 0{,}055$

Approx. changes by $0{,}05 \ {}^{m}/_{kg^2}$

6. (12pts) Use implicit differentiation to find $\frac{\partial z}{\partial y}$ at the point $(1, 1, 1)$, if $\sqrt{x} - \sqrt{y} + \sqrt{z} + \ln(xyz) = 1$.

$$F(x,y,z) = \sqrt{x} - \sqrt{y} + \sqrt{z} + \ln x + \ln y + \ln z$$

$$F_z = \frac{1}{2\sqrt{z}} + \frac{1}{z}$$

$$F_y = -\frac{1}{2\sqrt{y}} + \frac{1}{y}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{-\frac{1}{2\sqrt{y}} + \frac{1}{y}}{\frac{1}{2\sqrt{z}} + \frac{1}{z}}$$

$$\frac{\partial z}{\partial y}(1,1,1) = -\frac{-\frac{1}{2} + 1}{\frac{1}{2} + 1} = -\frac{\frac{1}{2}}{\frac{3}{2}} = -\frac{1}{3}$$

7. (20pts) Find and classify the local extremes for $f(x, y) = x^2 - xy^2 + y^2$.

$f_x = 2x - y^2$

$f_y = -2xy + 2y$

$\begin{cases} 2x - y^2 = 0 \\ 2y - 2xy = 0 \end{cases}$

$2y(1-x) = 0$

$y = 0$ or $x = 1$

case 1:

$y = 0$

$2x - 0^2 = 0$

$x = 0$

sol: $(0,0)$

case 2:

$x = 1$

$2 - y^2 = 0 \leftarrow$ plug into 1st eq

$y^2 = 2$

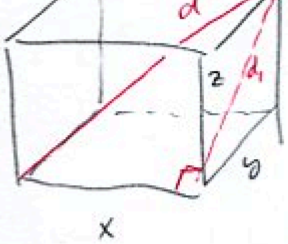
$y = \pm\sqrt{2}$

sol: $(1,\sqrt{2}), (1,-\sqrt{2})$

$D = \begin{vmatrix} 2 & -2y \\ -2y & 2-2x \end{vmatrix}$

(x,y)	$D(x,y)$
$(0,0)$	$\begin{vmatrix} \boxed{2} & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$ so local min ($2>0$)
$(1,\sqrt{2})$	$\begin{vmatrix} 2 & -2\sqrt{2} \\ -2\sqrt{2} & 0 \end{vmatrix} = 0 - 8 = -8$ saddle points
$(1,-\sqrt{2})$	$\begin{vmatrix} 2 & 2\sqrt{2} \\ 2\sqrt{2} & 0 \end{vmatrix} = 0 - 8 = -8$ saddle points

Bonus (10pts) Among all rectangular boxes of volume 1, find the one with the shortest diagonal d. *Hint: minimize* d^2.



$f(x,y) = d^2(x,y) = x^2 + y^2 + \left(\frac{1}{xy}\right)^2 = x^2 + y^2 + \frac{1}{x^2y^2} = x^2 + y^2 + x^{-2}y^{-2}$

$f_x = 2x - 2x^{-3}y^{-2} = 2x - \frac{2}{x^3y^2} \quad | \cdot x^3y^2$

$f_y = 2y - 2x^{-2}y^{-3} = 2y - \frac{2}{x^2y^3} \quad | \cdot x^2y^3$

$D = \begin{vmatrix} 1 + \frac{6}{x^4y^2} & \frac{4}{x^3y^3} \\ \frac{4}{x^3y^3} & 1 + \frac{6}{x^2y^4} \end{vmatrix}$

$D(1,1) = \begin{vmatrix} 7 & 4 \\ 4 & 7 \end{vmatrix} > 0$, $7 > 0$

$x^2 + d_1^2 = d^2$

$x^2 + y^2 + z^2 = d^2$

$xyz = 1$

so $z = \frac{1}{xy}$

$\begin{cases} 2x^4y^2 - 2 = 0 \\ 2x^2y^4 - 2 = 0 \end{cases}$

$\begin{cases} x^4y^2 = 1 \\ x^2y^4 = 1 \end{cases}$

$y^2 = \frac{1}{x^4}$, put in 2nd eq.

$x^2\left(\frac{1}{x^4}\right)^2 = 1$

$\frac{x^2}{x^8} = 1$

$\frac{1}{x^6} = 1$

$x^6 = 1$ so $x = 1$ which means $y = 1$

Box is a cube, $1 \times 1 \times 1$

This is a local min. at 1,1. Since there is only one critical point it's a absolute min.