Calculus 3 — Exam 1	
MAT 309, Spring 2018 — D. Ivanšić	

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1. (11pts) Let $\mathbf{u} = \langle 3, -7, 1\rangle$ and $\mathbf{v} = \langle 3, 0, -4\rangle$.
a) Calculate $3\mathbf{u}$, $2\mathbf{u} - \mathbf{v}$, and $\mathbf{u}\cdot\mathbf{v}$.
b) Find a vector of length 4 in direction of $\mathbf{u}$.
c) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

a) $3\vec{u} = \langle 9, -21, 3\rangle$

$2\vec{u} - \vec{v} = \langle 6, -14, 2\rangle - \langle 3, 0, -4\rangle$

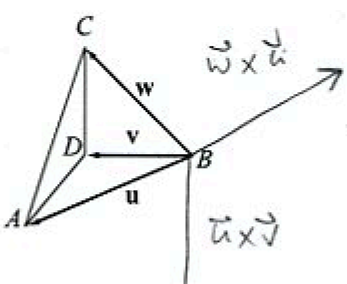
$= \langle 3, -14, 6\rangle$

$\vec{u}\cdot\vec{v} = 9 + 0 - 4 = 5$

b) $\dfrac{4}{|\vec{u}|}\vec{u} = \dfrac{4}{\sqrt{9+49+1}}\langle 3, -7, 1\rangle = \dfrac{4}{\sqrt{59}}\langle 3, -7, 1\rangle$

c) $\mathrm{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \dfrac{5}{9+16}\langle 3, 0, -4\rangle = \dfrac{1}{5}\langle 3, 0, -4\rangle$

2. (12pts) In the picture, a tetrahedron $ABCD$ is given. All edges that meet at D are perpendicular to each other and have length 4. Note that this makes ABC an equilateral triangle. Draw the vectors $\mathbf{u}\times\mathbf{v}$ and $\mathbf{w}\times\mathbf{u}$ and determine their length.



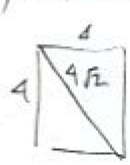
$\vec{u}\times\vec{v}$ is in opposite direction of $\vec{DC}$

$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin 45^\circ = 4\sqrt{2}\,4\cdot\frac{\sqrt{2}}{2} = 16$

$\vec{w}\times\vec{u}$ is perp. to triangle ABC

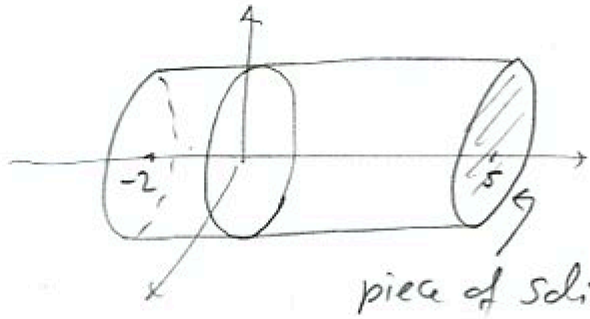
$|\vec{w}\times\vec{u}| = |\vec{w}||\vec{u}|\sin 60^\circ = 4\sqrt{2}\,4\sqrt{2}\cdot\frac{\sqrt{3}}{2} = 16\sqrt{3}$

$|\vec{u}|$, $|\vec{w}|$ are lengths of diagonals in a square.



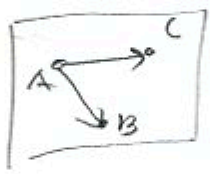
3. (8pts) Draw the region in $\mathbf{R}^3$ described by:

$\underbrace{x^2 + z^2 \le 4}_{\text{solid cylinder}}, \; -2 \le y \le 5$



piece of solid cylinder of radius 2

4. (12pts) Find the equation of the plane that contains the points $A = (0,3,-4)$, $B = (3,1,-2)$ and $C = (4,0,2)$.



$\vec{AB} = \langle 3,-2,2\rangle$

$\vec{AC} = \langle 4,-3,6\rangle$

$$\vec{n} = \vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 2 \\ 4 & -3 & 6 \end{vmatrix} = (-12+6)\vec{i} - (18-8)\vec{j} + (-9+8)\vec{k}$$

$$= -6\vec{i} - 10\vec{j} - \vec{k}$$

Take $\vec{n} = 6\vec{i} + 10\vec{j} + \vec{k}$

$$6(x-0) + 10(y-3) + 1\cdot(z+4) = 0$$

$$6x + 10y + z = 26$$

5. (16pts) This problem is about the surface $\dfrac{x^2}{4} - \dfrac{y^2}{9} - \dfrac{z^2}{1} = 1$.

a) Identify and sketch the intersections of this surface with the coordinate planes.

b) Sketch the surface in 3D, with coordinate system visible.

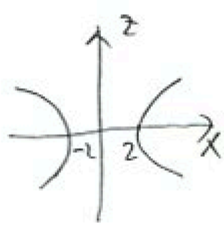
a) $x = 0$

$\underbrace{-\frac{y^2}{9} - \frac{z^2}{1}}_{<0} = 1 > 0$ nothing

$y = 0$

$\frac{x^2}{4} - \frac{z^2}{1} = 1$

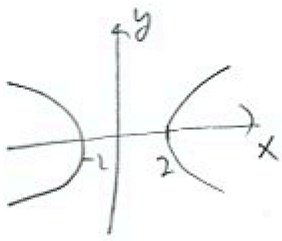
hyperbola

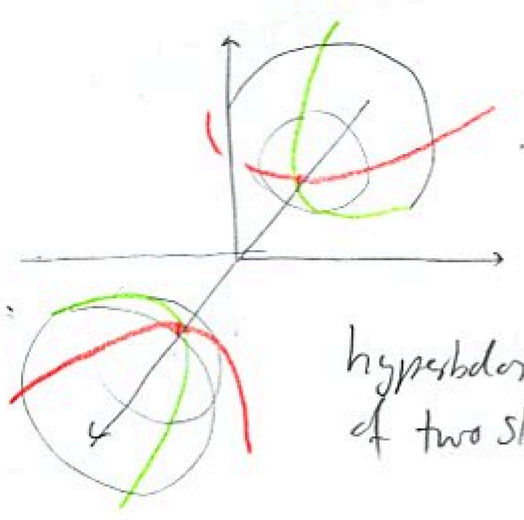


$z = 0$

$\frac{x^2}{4} - \frac{y^2}{9} = 1$

hyperbola





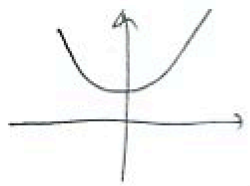
hyperboloid of two sheets

6. (14pts) The curve $\mathbf{r}(t) = \langle t, t^2+1, \sin t\rangle$ is given, t any real number.

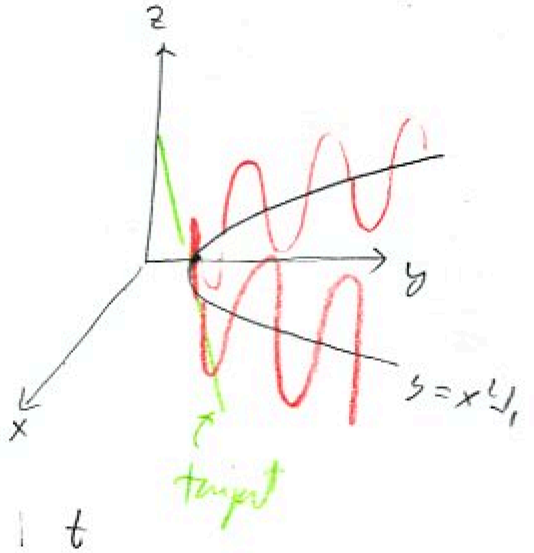
a) Sketch the curve in the coordinate system.

b) Find parametric equations of the tangent line to this curve when $t = 0$ and sketch the tangent line.

a) In xy plane it traces out $y = x^2+1$



In z direction it oscillates between -1 and 1



b) $\vec{r}\,'(t) = \langle 1, 2t, \cos t\rangle$

$\vec{r}\,'(0) = \langle 1, 0, 1\rangle$

$\vec{r}(0) = \langle 0, 1, 0\rangle$

$x = t$

$y = 1$

$z = t$

7. (13pts) Find the length of the curve $\mathbf{r}(t) = \left\langle t\cos t, t\sin t, \frac{2\sqrt{2}}{3}t^{\frac{3}{2}}\right\rangle$, $0 \le t \le 4$.

$$\vec{r}\,'(t) = \left\langle \cos t - t\sin t,\ \sin t + t\cos t,\ \frac{2\sqrt{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}t^{\frac{1}{2}}\right\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + (\sqrt{2t})^2}$$

$$= \sqrt{\cos^2 t - \cancel{2t\cos t\sin t} + t^2\sin^2 t + \sin^2 t + 2t\cos t\sin t + t^2\cos^2 t + 2t}$$

$= 1$ $\quad$ $= t^2$

$$= \sqrt{1 + t^2 + 2t}$$

$$= \sqrt{(1+t)^2} = |1+t| = 1+t, \text{ since } t \ge 0, \text{ so } 1+t \ge 0$$

$$\int_0^4 (1+t)\,dt = 1\cdot(4-0) + \frac{t^2}{2}\Big|_0^4 = 4 + \frac{1}{2}(16-0) = 12$$

8. (14pts) Suppose the corner of a room is represented by the three coordinate planes. An egg is launched from point $(3, 7, 6)$ with initial velocity vector $\mathbf{v_0} = \langle -5, -2, 7 \rangle$.
a) Assuming gravity acts in the usual negative z-direction (let $g = 10$), find the vector function $\mathbf{r}(t)$ representing the egg's position.
b) On which wall or floor does the egg splatter?

a) $\vec{a}(t) = \langle 0, 0, -10 \rangle$

$\vec{v}(t) = \langle 0, 0, -10t \rangle + \vec{C}$

$\langle -5, -2, 7 \rangle = \vec{v}(0) = \vec{0} + \vec{C}$

$\vec{C} = \langle -5, -2, 7 \rangle$

$\vec{v}(t) = \langle -5, -2, -10t + 7 \rangle$

$\vec{r}(t) = \langle -5t, -2t, -5t^2 + 7t \rangle + \vec{D}$

$\langle 3, 7, 6 \rangle = \vec{r}(0) = \vec{0} + \vec{D}$

$\vec{D} = \langle 3, 7, 6 \rangle$

$\vec{r}(t) = \langle 3 - 5t, 7 - 2t, -5t^2 + 7t + 6 \rangle$

b) Trajectory crosses coordinate planes when

$x = 0$ .. $3 - 5t = 0$, $t = \frac{5}{3}$

$y = 0$ .. $7 - 2t = 0$, $t = \frac{7}{2}$

$z = 0$ $5t^2 - 7t - 6 = 0$

$t = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 5 \cdot (-6)}}{25}$

$= \frac{7 \pm \sqrt{49 + 120}}{10} = \frac{7 \pm 13}{10} = 2, -\frac{3}{5}$ not valid

Since $t = \frac{5}{3}$ is smallest we conclude the egg hits yz-plane first.

Bonus (10pts) For every pair of skew (nonintersecting) lines in $\mathbf{R}^3$, there is a line that intersects them both and is perpendicular to both. Find this line for the two lines given parametrically below. *Hint: if P is a point on one of the lines, and Q on the other, what conditions have to be satisfied so that the line determined by P and Q is the requested line?*

$$\underbrace{\begin{aligned} x &= -8 + 5t \\ y &= -1 + t \\ z &= -5 + 2t \end{aligned}}_{\text{generic } P} \qquad \underbrace{\begin{aligned} x &= 3 + 2s \\ y &= -4 - 2s \\ z &= 2 - s \end{aligned}}_{Q \text{ a line}}$$

We must have

$\vec{PQ} \perp \vec{v}_1, \vec{v}_2$, i.e. $\vec{PQ} \cdot \vec{v}_1 = 0$, $\vec{PQ} \cdot \vec{v}_2 = 0$.

$\vec{PQ} = \langle 11 + 2s - 5t, -3 - 2s - t, 7 - s - 2t \rangle$

$\cdot \langle 5, 1, 2 \rangle = 0$

$\cdot \langle 2, -2, -1 \rangle = 0$

$\vec{PQ} \cdot \vec{v}_1 = 55 + 10s - 25t - 3 - 2s - t + 14 - 2s - 4t = 0$

$\vec{PQ} \cdot \vec{v}_2 = 22 + 4s - 10t + 6 + 4s + 2t - 7 + s + 2t = 0$

$$\begin{cases} 66 + 6s - 30t = 0 \\ 21 + 9s - 6t = 0 \end{cases}$$

$30t - 6s = 66$

$6t - 9s = 21 \quad \cdot(-5)$

$39s = -105 + 66$

$39s = -39 \quad s = -1, t = 2$

$30t = 60$

$t = 2$

$P = (2, 1, -1)$, $Q = (1, -2, 3)$ $\quad \vec{PQ} = \langle -1, -3, 4 \rangle$

take $\vec{v} = \langle 1, 3, -4 \rangle$

line PQ: $x = 2 + u$, $y = 1 + 3u$, $z = -1 - 4u$