Calculus 1 — Final Exam
MAT 250, Spring 2017 — D. Ivanšić

Name: Saul Ocean

Show all your work!

1. (15pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

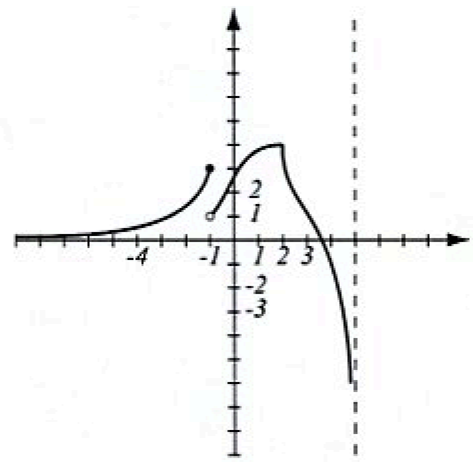
$\lim_{x \to -1^-} f(x) = 3$

$\lim_{x \to -1^+} f(x) = 1$

$\lim_{x \to -1} f(x) =$ dne, one-sided limits are not same

$\lim_{x \to 5^-} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = 0$



List points where f is not continuous and explain why.

$x = -1$, $\lim_{x \to -1} f(x)$ doesn't exist

List points where f is not differentiable and explain why.

$x = -1$ not even continuous there, so not diff

$x = 2$ sharp point

Find the following limits algebraically. Do not use L'Hospital's rule.

2. (6pts) $\lim_{x \to \infty} \frac{x^2 - 3x + 4}{3x^2 - 4x - 2} = \lim_{x \to \infty} \frac{\cancel{x^2}\left(1 - \frac{3}{x} + \frac{4}{x^2}\right)}{\cancel{x^2}\left(3 - \underbrace{\frac{4}{x}}_{\to 0} - \underbrace{\frac{2}{x^2}}_{\to 0}\right)} = \frac{1 - 0 + 0}{3 - 0 - 0} = \frac{1}{3}$

3. (6pts) $\lim_{x \to 3^-} \frac{x^2 + 5x}{2x - 6} = \frac{24}{0^-} = -\infty$ $\left(\text{or } \frac{24}{\text{small neg}} = \text{large neg.}\right)$

when $x < 3$

$2x - 6 < 0$

$-$ 3 $+$

4. (6pts) The equation $x^2 + \cos x = 5 - 2x$ is given. Use the Intermediate Value Theorem to show it has a solution.

$$\underbrace{x^2+2x-5+\cos x}_{f(x)} = 0$$

f is continuous on $\mathbb{R}$

$f(0) = -5 + \cos 0 = -4$

$f(3) = 9+6-5+\cos 3 = 10 + \cos 3 > 0$

since $-1 < \cos 3 < 1$

Since $f(0) < 0 < f(3)$, by the Intermediate Value Theorem there is a point c in $(0,3)$ s.t. $f(c) = 0$

5. (14pts) The curve with equation $\sin x \sin y = x^2 - y^2$ is given.
a) Use implicit differentiation to find y'.
b) Find the equation of the tangent line at (π, π).

a) $\sin x \sin y = x^2 - y^2 \quad \Big| \frac{d}{dx}$

$\cos x \sin y + \sin x \cos y \cdot y' = 2x - 2yy'$

$\sin x \cos y\, y' + 2yy' = 2x - \cos x \sin y$

$y'(\sin x \cos y + 2y) = 2x - \cos x \sin y$

$$y' = \frac{2x - \cos x \sin y}{\sin x \cos y + 2y}$$

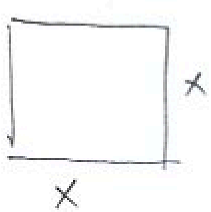
b) When $x = \pi$, $y = \pi$

$$y' = \frac{2\pi - \cos\pi \overset{0}{\sin\pi}}{\underset{0}{\sin\pi}\cos\pi + 2\pi} = 1$$

$y - \pi = 1 \cdot (x - \pi)$

$y = x$ Eq. of tan. line

6. (8pts) A square field is measured to have side length of 5 kilometers, with maximum error 25 meters. Use differentials to estimate the maximum possible error, the relative error and the percentage error when computing the area of the field.



$A = x^2$

$dA = 2x\,dx$

$x = 5$ km

$dx = \frac{25}{1000}$ km

$dA = 2 \cdot 5 \cdot \frac{25}{1000} = \frac{250}{1000} = \frac{1}{4}$ km^2

relative error $= \frac{\frac{1}{4}}{5^2} = \frac{1}{4 \cdot 25} = \frac{1}{100} = 1\%$

(percentage error)

7. (26pts) Let $f(x) = (x^2+1)e^x$. Draw an accurate graph of f by following the guidelines.
a) Find the intervals of increase and decrease, and local extremes.
b) Find the intervals of concavity and points of inflection.
c) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Use L'Hospital's rule where necessary.
d) Use information from a)–c) to sketch the graph.

a) $f'(x) = 2xe^x + (x^2+1)e^x$

$= (x^2+2x+1)e^x$

crit. pts: $(x^2+2x+1)e^x = 0$

$(x+1)^2 \underbrace{e^x}_{>0 \text{ always}} = 0$

$x = -1$

Since $(x+1)^2 \geq 0$, $e^x > 0$, $f'(x) \geq 0$,

so f is always increasing

b) $f''(x) = (2x+2)e^x + (x^2+2x+1)e^x$

$= (x^2+4x+3)e^x$

$(x^2+4x+3)\underbrace{e^x}_{>0} = 0$

$(x+1)(x+3) = 0$, $x = -3, -1$

$(x+1)(x+3)$

		-3		-1	
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	CU	IP	CD	IP	CU

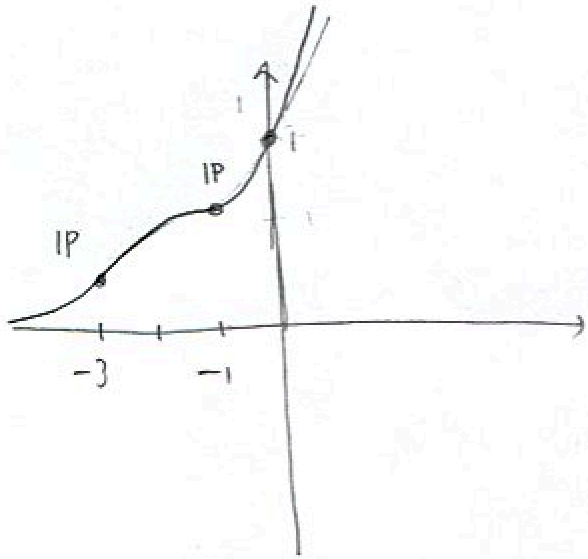
c) $\lim_{x\to\infty} (x^2+1)e^x = (\infty^2+1)e^{\infty} = \infty \cdot \infty = \infty$

$\lim_{x\to-\infty} \underbrace{(x^2+1)e^x}_{\infty \cdot 0} = \lim_{x\to-\infty} \dfrac{\overset{\to\infty}{x^2+1}}{\underset{\to\infty}{e^{-x}}} \overset{L'H}{=}$

$= \lim_{x\to-\infty} \dfrac{\overset{\to-\infty}{2x}}{\underset{\to-\infty}{-e^{-x}}} \overset{L'H}{=} \lim_{x\to-\infty} \dfrac{2}{-e^{-x}\cdot(-1)} = \dfrac{2}{\infty} = 0$

d)

x	$(x^2+1)e^x$
-1	$2e^{-1} = \frac{2}{e} \approx \frac{2}{3}$
-3	$10e^{-3} = \frac{10}{e^3} \approx \frac{10}{27}$



8. (6pts) Find $f(x)$ if $f'(x) = \sqrt[3]{x^4} + \dfrac{1}{x}$ and $f(1) = 5$.

$$f'(x) = x^{\frac{4}{3}} + \frac{1}{x}$$

$$f(x) = \frac{x^{\frac{7}{3}}}{\frac{7}{3}} + \ln|x| = \frac{3}{7}x^{\frac{7}{3}} + \ln|x| + C$$

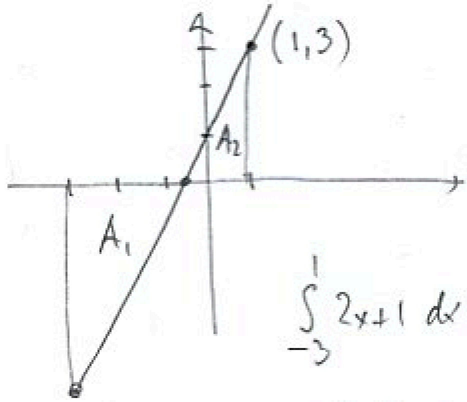
$$5 = f(1) = \frac{3}{7}\cdot 1^{\frac{7}{3}} + \ln 1 + C$$

$$C = 5 - \frac{3}{7} = \frac{32}{7}$$

$$f(x) = \frac{3}{7}x^{\frac{7}{3}} + \ln|x| + \frac{32}{7}$$

9. (13pts) Find $\int_{-3}^{1} 2x + 1\,dx$ in two ways (they'd better give you the same answer!):
a) Using the "area" interpretation of the integral. Draw a picture.
b) Using the Evaluation Theorem.

a)



$$\int_{-3}^{1} 2x+1\,dx = -A_1 + A_2$$

$$= -\frac{1}{2}\cdot\frac{5}{2}\cdot 5 + \frac{1}{2}\cdot\frac{3}{2}\cdot 3$$

$$= \frac{9-25}{4} = -\frac{16}{4} = -4$$

b) $\displaystyle\int_{-3}^{1} 2x+1\,dx$

$$= (x^2+x)\Big|_{-3}^{1} = 1^2-(-3)^2+(1-(-3))$$

$$= 1-9+1+4 =$$

Use the substitution rule in the following integrals:

10. (7pts) $\displaystyle\int (e^x + 1)\sec^2(e^x + x)\,dx =$ $\left[\begin{array}{l} u = e^x + x \\ du = e^x + 1\,dx \end{array}\right] = \int \sec^2 u\,du$

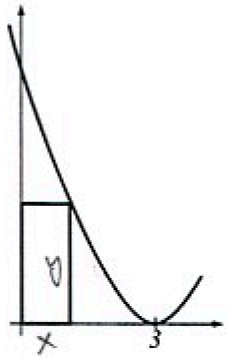
$$= \tan u = \tan(e^x + x) + C$$

11. (9pts) $\displaystyle\int_1^3 \frac{6x+3}{\sqrt{x^2+x+4}}\,dx =$ $\left[\begin{array}{ll} u = x^2+x+4 & x=3,\ u=16 \\ du = 2x+1\,dx & x=1,\ u=6 \\ 3du = 6x+3\,dx & \end{array}\right] = \displaystyle\int_6^{16} \frac{1}{\sqrt{u}}\,3du$

$u^{-\frac{1}{2}}$

$$= \frac{3u^{\frac{1}{2}}}{\frac{1}{2}}\Big|_6^{16} = 6\sqrt{u}\Big|_6^{16} = 6(\sqrt{16}-\sqrt{6}) = 6(4-\sqrt{6})$$

12. (18pts) Consider a rectangle with sides on the x- and y-axes whose one vertex lies on the parabola $y = (x-3)^2$ and is enclosed in the region between the axes and the parabola. Among all such rectangles, find the one with the biggest area.



$A = xy = x(x-3)^2$

Job: maximize $A(x)$ on $[0,3]$

$A'(x) = 1\cdot(x-3)^2 + x\cdot 2(x-3)$

$= (x-3)(x-3+2x)$

$= (x-3)(3x-3)$

$= 3(x-3)(x-1)$

crit. pts: $A'(x)=0$

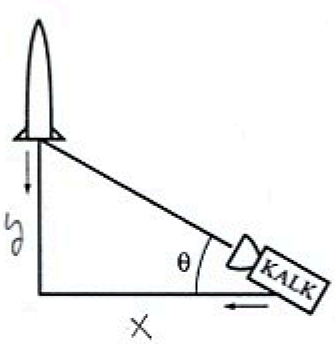
$x = 1, 3$

Closed interval method:

x	$x(x-3)^2$
0	0
3	0
1	4 ← max

Area is greatest for $x = 1$

13. (16pts) Late to the event, the crew of TV-station KALK is rushing on a straight road to a vertical landing of a new rocket. Their rooftop camera is aimed at the rocket. KALK's car is approaching the landing site at 20 meters per second and the rocket, located directly above the landing site, is descending at 12 meters per second. At what rate is the angle of elevation θ of the line of sight to the rocket changing when the rocket is 200 meters above ground, and the car is 1000 meters from the landing site? Is the camera tilting lower or higher at that moment? *(Hint: sohcahtoa.)*



Know: $y' = -12$ m/s

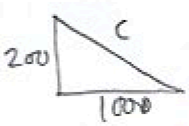
$x' = -20$ m/s

$\tan\theta = \frac{y}{x} \quad \Big| \frac{d}{dt}$

$\sec^2\theta \cdot \theta' = \frac{y'x - yx'}{x^2}$

$\theta' = \frac{y'x - yx'}{\sec^2\theta \cdot x^2} = \boxed{\frac{y'x - yx'}{x^2}\cos^2\theta}$

When $x = 1000$, $y = 200$



$c^2 = 200^2 + 1000^2$

$c^2 = 40000 + 1{,}000{,}000$

$c^2 = 1{,}040{,}000$

$\cos^2\theta = \frac{1000000}{1040000} = \frac{100}{104} = \frac{25}{26}$

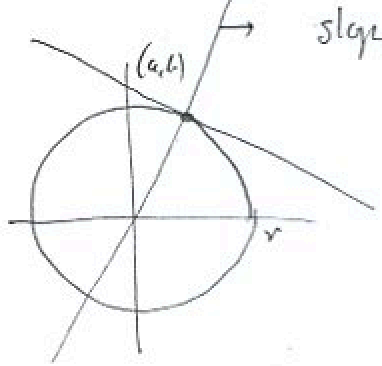
$\theta' = \frac{-12\cdot 1000 - 200\cdot(-20)}{1000^2}\cdot\frac{25}{26}$

$= \frac{4000 - 12000}{1000000}\cdot\frac{25}{26} = \cancel{\frac{2000}{1000000}\cdot\frac{25}{26} = \frac{7}{2600}}$ rad/s

$-\frac{1}{130}$ rad/s $= \frac{-\cancel{8000}}{\cancel{1000000}}\cdot\frac{\cancel{25}}{26}$

Since $\theta' < 0$, camera is tilting lower.

Bonus. (8pts) It is common knowledge that the tangent line to a circle is perpendicular to the radius at the point of tangency. Show this fact using slopes of the tangent line and the radius line through point (a, b) of the circle $x^2 + y^2 = r^2$. *(Hint: implicit differentiation will make it easier here.)*



slope $= \frac{b-0}{a-0} = \frac{b}{a}$

$x^2 + y^2 = r^2 \quad \Big| \frac{d}{dx}$

$2x + 2yy' = 0$

$y' = -\frac{2x}{2y} = -\frac{x}{y}$

At (a,b), $y' = -\frac{a}{b}$, the opposite reciprocal of $\frac{b}{a}$,

so the lines are perpendicular,

Bonus. (7pts) Find the limit algebraically.

$$\lim_{x\to\infty} (\sqrt{x^2+5x+2} - x) = \lim_{x\to\infty} \left(\sqrt{x^2+5x+2} - x\right) \frac{\sqrt{x^2+5x+2}+x}{\sqrt{x^2+5x+2}+x}$$

$$= \lim_{x\to\infty} \frac{x^2+5x+2-x^2}{\sqrt{x^2+5x+2}+x} = \lim_{x\to\infty} \frac{5x+2}{\sqrt{x^2\left(1+\frac{5}{x}+\frac{2}{x^2}\right)}+x}$$

$x\sqrt{\ }$

$$= \lim_{x\to\infty} \frac{x\left(5+\frac{2}{x}\right)}{x\left(\sqrt{1+\frac{5}{x}+\frac{2}{x^2}}+1\right)} = \frac{5+0}{\sqrt{1+0+0}+1} = \frac{5}{2}$$