Calculus 1 — Exam 3	**Name:** Saul Ocean
MAT 250, Spring 2017 — D. Ivanšić	*Show all your work!*

Differentiate and simplify where appropriate:

1. (~~4~~ 5pts) $\dfrac{d}{dx}(x^2+4x-5)e^x = (2x+4)e^x+(x^2+4x-5)e^x$

$$= (x^2+6x-1)e^x$$

2. (4pts) $\dfrac{d}{dx}\ln(\sqrt{\cos x+1}) = \dfrac{d}{dx}\,\dfrac{1}{2}\ln(\cos x+1) = \dfrac{1}{2}\,\dfrac{1}{\cos x+1}\cdot(-\sin x)$

$(\cos x+1)^{\frac{1}{2}}$

$$= -\frac{\sin x}{2(\cos x+1)}$$

3. (~~6~~ 7pts) $\dfrac{d}{dt}\,\dfrac{2^t+t^2}{t} = \dfrac{(\ln 2\cdot 2^t+2t)\cdot t-(2^t+t^2)\cdot 1}{t^2} = \dfrac{(\ln 2\cdot t-1)2^t+t^2}{t^2}$

4. (7pts) $\dfrac{d}{dx}\ln\dfrac{1+\sin x}{1-\sin x} = \dfrac{d}{dx}\Big(\ln(1+\sin x)-\ln(1-\sin x)\Big) = \dfrac{1}{1+\sin x}\cdot\cos x-\dfrac{1}{1-\sin x}(-\cos x)$

$$= \cos x\left(\frac{1}{1+\sin x}+\frac{1}{1-\sin x}\right) = \cos x\,\frac{1-\sin x+1+\sin x}{1-\sin^2 x}$$

$$= \frac{2\cos x}{\cos^2 x} = \frac{2}{\cos x} = 2\sec x$$

5. (7pts) $\dfrac{d}{dw}\,\dfrac{\arccos w}{\sqrt{1-w^2}} = \dfrac{-\frac{1}{\sqrt{1-w^2}}\sqrt{1-w^2}-\arccos w\cdot\frac{1}{2\sqrt{1-w^2}}\cdot(-2w)}{\left(\sqrt{1-w^2}\right)^2}$

$$= \frac{-1+\frac{w\arccos w}{\sqrt{1-w^2}}}{1-w^2}\cdot\frac{\sqrt{1-w^2}}{\sqrt{1-w^2}} = \frac{w\arccos w-\sqrt{1-w^2}}{(1-w^2)^{3/2}}$$

6. (10pts) Use logarithmic differentiation to find the derivative of $y=(\cos x)^{\sin x}$.

$$y=(\cos x)^{\sin x}$$

$$\ln y = \ln(\cos x)^{\sin x} = \sin x\cdot\ln\cos x \;\Big|\frac{d}{dx}$$

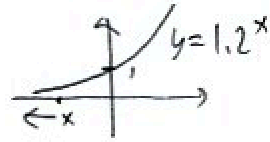
$$\frac{1}{y}\cdot y' = \cos x\ln\cos x+\sin x\cdot\frac{1}{\cos x}\cdot(-\sin x)$$

$$y' = y(\quad) = (\cos x)^{\sin x}(\cos x\ln\cos x-\sin x\tan x)$$

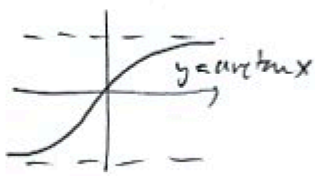
Find the limits algebraically. Graphs of basic functions will help, as will L'Hospital's rule, where appropriate.

7. (2pts) $\lim\limits_{x\to-\infty} 1.2^x = \boxed{\infty}$

since $1.2 > 1$



8. (6pts) $\lim\limits_{x\to\infty} \arctan\left(\dfrac{x^2+4}{x+1}\right) = \arctan\left(\lim\limits_{x\to\infty} \dfrac{x^2+4}{x+1}\right) = \arctan\left(\lim\limits_{x\to\infty} \dfrac{x^{\cancel{2}}\left(1+\overbrace{\frac{4}{x^2}}^{\to 0}\right)}{\cancel{x}\left(1+\underbrace{\frac{1}{x}}_{\to 0}\right)}\right)$



$= \arctan\left(\lim\limits_{x\to\infty} x\cdot 1\right) = \arctan \infty = \boxed{\dfrac{\pi}{2}}$

9. (7pts) $\lim\limits_{x\to0} \dfrac{\overbrace{\ln(x+1)-x}^{\ln 1 - 0 = 0 - 0 = 0}}{\underbrace{x^2}_{\to 0}} \overset{L'H}{=} \lim\limits_{x\to0} \dfrac{\overbrace{\frac{1}{x+1}-1}^{\to \frac{1}{1}-1=0}}{\underbrace{2x}_{\to 0}} \overset{L'H}{=} \lim\limits_{x\to0} \dfrac{-\frac{1}{(x+1)^2}\cdot 1}{2} = \lim\limits_{x\to0} -\dfrac{1}{2(x+1)^2}$

$\boxed{= -\dfrac{1}{2}}$

10. (8pts) $\lim\limits_{x\to\infty} x^2 \sin\dfrac{1}{x} = \lim\limits_{x\to\infty} \dfrac{\overbrace{\sin\frac{1}{x}}^{\to \sin 0 = 0}}{\underbrace{\frac{1}{x^2}}_{\to 0^+}} \overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{\cos\frac{1}{x}\cdot\left(\cancel{-}\frac{1}{x^2}\right)}{\cancel{-}\frac{2}{x^3}} = \lim\limits_{x\to\infty} \cos\dfrac{1}{x} \cdot \lim\limits_{x\to\infty} \dfrac{1}{x^2}\cdot\dfrac{x^3}{2}$

$\infty \cdot \sin\frac{1}{\infty}$

$\infty \cdot 0$

$(x^{-2})' = -2x^{-3}$

$= \cos 0 \cdot \lim\limits_{x\to\infty} \dfrac{x}{2} = 1\cdot\dfrac{\infty}{2} = \boxed{\infty}$

11. (9pts) $\lim\limits_{x\to\infty} (e^x+x)^{\frac{1}{x}} = \lim\limits_{x\to\infty} e^{\ln y} = e^1 = \boxed{e}$

$(\infty+\infty)^{\frac{1}{\infty}} = \infty^0$

$y = (e^x+x)^{\frac{1}{x}}$

$\ln y = \ln(e^x+x)^{\frac{1}{x}} = \dfrac{\ln(e^x+x)}{x}$

$\lim\limits_{x\to\infty} \dfrac{\overbrace{\ln(e^x+x)}^{\ln(\infty+\infty)=\infty}}{\underbrace{x}_{\to\infty}} \overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{\frac{1}{e^x+x}\cdot(e^x+1)}{1} =$

$= \lim\limits_{x\to\infty} \dfrac{\overbrace{e^x+1}^{\to\infty}}{\underbrace{e^x+x}_{\to\infty}} \overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{e^x} = 1$

12. (10pts) Let $f(x) = e^x$.
a) Write the linearization of $f(x)$ at $a = 0$.
b) Use the linearization to estimate $e^{0.15}$ and compare to the calculator value of 1.161834.

a) $f(0) = e^0 = 1 \quad (e^x)' = e^x$

$f'(0) = e^0 = 1$

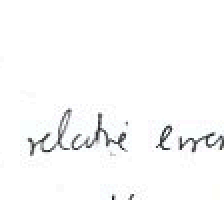
$L(x) = f(0) + f'(0)(x-0)$

$= 1 + 1\cdot(x-0)$

$= 1+x$

b) $e^{0.15} \approx L(0.15) = 1 + 0.15 = 1.15$,
within 0.012 of actual value

13. (10pts) A square field is measured to have side length of 16 kilometers, with maximum error 40 meters. Use differentials to estimate the maximum possible error, the relative error and the percentage error when computing the area of the field.



$A = x^2$

$dA = 2x\,dx$

$$dA = 2\cdot 16 \cdot \frac{4\cancel{0}}{100\cancel{0}} = \frac{8\cdot \overset{4}{\cancel{16}}}{\underset{25}{\cancel{100}}} = \frac{32}{25} = 1.28 \text{ km}^2$$

relative error,

$$\frac{dA}{A} = \frac{\frac{32}{25}}{16^2} = \frac{\overset{2}{\cancel{32}}}{25\cdot 16^{\cancel{2}}} = \frac{\overset{1}{\cancel{2}}}{25\cdot\cancel{16}\,8} = \frac{1}{200} = 0.5\%$$

14.
~~15~~. (8pts) Let $f(x) = x^3 + 4x^2 - 7x + 6$. Use the theorem on derivatives of inverses to find $(f^{-1})'(4)$.

$f'(x) = 3x^2 + 8x - 7$

$f^{-1}(4)$ is solution to

$f(x) = 4$

$x^3 + 4x^2 - 7x + 6 = 4$

$x^3 + 4x^2 - 7x = -2$

guess: $x = 1$

$f^{-1}(4) = 1$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(1)}$$

$$= \frac{1}{3+8-7} = \frac{1}{4}$$

Bonus. (10pts) We have stated that 0^0 is an indeterminate form, yet most limits of this type come out equal to 1. The purpose of this problem is to find examples where $0^0 \neq 1$.
a) Show that $\lim_{x\to 0} e^{-\frac{1}{x^2}} = 0$.
b) Find three different functions $g(x)$ so that $\lim_{x\to 0} g(x) = 0$ in each case, but $\lim_{x\to 0}(e^{-\frac{1}{x^2}})^{g(x)}$ equals $\frac{1}{e}$, 0, and $\frac{1}{2}$, respectively. Thus, by setting $f(x) = e^{-\frac{1}{x^2}}$, you will get three limits $\lim_{x\to 0} f(x)^{g(x)}$ of type 0^0 that do not equal 1.

a) $\lim_{x\to 0} e^{-\frac{1}{x^2}} = e^{-\frac{1}{0+}} = e^{-\infty} = 0$

b) $\left(e^{-\frac{1}{x^2}}\right)^{?} \to e^{-1}$ $\quad\left(e^{-\frac{1}{x^2}}\right)^{?} \to 0$ $\quad\left(e^{-\frac{1}{x^2}}\right)^{?} \to \frac{1}{2}$

$\lim_{x\to 0}\left(e^{-\frac{1}{x^2}}\right)^{x^2} = \lim_{x\to 0} e^{-1} = e^{-1}$

$\left(\lim_{x\to 0} x^2 = 0\right)$

$\lim_{x\to 0}\left(e^{-\frac{1}{x^2}}\right)^{|x|} = \lim_{x\to 0} e^{-\frac{1}{|x|}} = e^{-\frac{1}{0+}}$

$= e^{-\infty} = 0$

$\left(\lim_{x\to 0} |x| = 0\right)$

$\lim_{x\to 0}\left(e^{-\frac{1}{x^2}}\right)^{x^2 \cdot \ln 2} = \lim_{x\to 0} e^{-\ln 2} = e^{-\ln 2}$

$= \frac{1}{e^{\ln 2}} = \frac{1}{2}$ $\quad\left(\lim_{x\to 0} x^2 \cdot \ln 2 = 0\right)$