Calculus 1 — Exam 2
MAT 250, Spring 2017 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\frac{d}{dx}\left(2x^4+\frac{8}{x^5}-\frac{1}{\sqrt[9]{x^4}}-\pi^5\right)=$ $8x^3-40x^{-6}+\frac{4}{9}x^{-\frac{13}{9}}$

$\frac{d}{dx}\left(2x^4+8x^{-5}-x^{-\frac{4}{9}}-\text{const}\right)$

2. (5pts) $\frac{d}{du}\sin u\tan u=$ $\cos u\tan u+\sin u\sec^2 u=\sin u+\sin u\sec^2 u$

$\frac{\sin u}{\cos u}$

3. (6pts) $\frac{d}{dx}\frac{3x-1}{x^3+2x^2+1}=$ $\frac{3(x^3+2x^2+1)-(3x-1)(3x^2+4x)}{(x^3+2x^2+1)^2}$

$=\frac{3x^3+6x^2+3-(9x^3-3x^2+12x^2-4x)}{(x^3+2x^2+1)^2}=\frac{-6x^3-3x^2+4x+3}{(x^3+2x^2+1)^2}$

4. (6pts) $\frac{d}{d\theta}(\cos^2\theta-\cos(2\theta))=$ $\underbrace{2\cos\theta(-\sin\theta)}-(-\sin(2\theta))\cdot 2$

$=-\sin(2\theta)+\sin(2\theta)=0$

5. (6pts) $\frac{d}{dx}\sin\sqrt{x^2+3x+5}=$ $\cos\sqrt{x^2+3x+5}\cdot\frac{1}{2\sqrt{x^2+3x+5}}\cdot(2x+3)$

$=\frac{(2x+3)\cos\sqrt{x^2+3x+5}}{2\sqrt{x^2+3x+5}}$

6. (7pts) The limit at right represents a derivative $f'(a)$.
a) State f and a.
b) Evaluate $f'(a)$ using differentiation rules — this gives you the limit.

$\lim_{x\to 2}\frac{x^5-32}{x-2}$ 2^5

b) $f'(x)=5x^4$

$f'(2)=5\cdot 2^4=5\cdot 16=80$

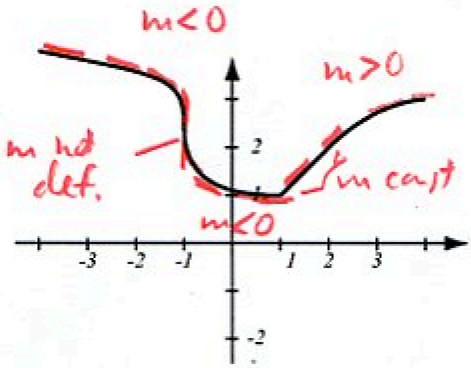
a) $=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

so $a=2$

$f(x)=x^5$

7. (10pts) The graph of the function $f(x)$ is shown at right.

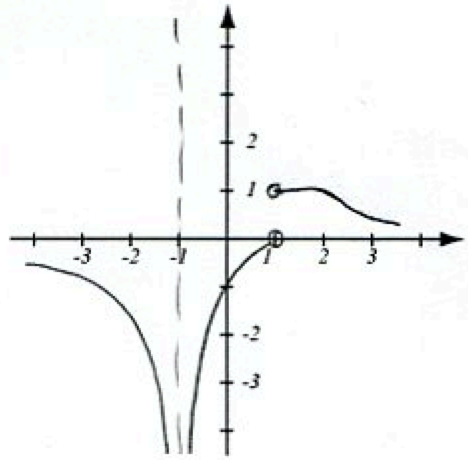
a) Where is $f(x)$ not differentiable? Why?
b) Use the graph of $f(x)$ to draw an accurate graph of $f'(x)$.



a) at $x = -1$ vertical tangent line

at $x = 1$ sharp point

b)



8. (14pts) Let $f(x) = \frac{4}{x}$.
a) Use the limit definition of the derivative to find the derivative of the function.
b) Check your answer by taking the derivative of f using differentiation rules.
c) Write the equation of the tangent line to the curve $y = f(x)$ at point $(1, 4)$.

a) $f'(a) = \lim\limits_{x\to a} \dfrac{\frac{4}{x} - \frac{4}{a}}{x-a} = \lim\limits_{x\to a} \dfrac{\frac{4a-4x}{xa}}{x-a} = \lim\limits_{x\to a} \dfrac{4(a-x)}{(x-a)\,xa}$

$= \lim\limits_{x\to a} \dfrac{4\cancel{(a-x)}^{-1}}{\cancel{(x-a)}\,xa} = \lim\limits_{x\to a} \dfrac{-4}{xa} = -\dfrac{4}{a^2}$

b) $f'(x) = (4x^{-1})' = -4x^{-2} = -\frac{4}{x^2}$, agrees

c) $f'(1) = -\frac{4}{1^2} = -4$ Eq: $y - 4 = -4(x-1)$

$y = -4x + 8$

9. (8pts) Let $g(x) = \dfrac{x}{f(x)}$ and $h(x) = f(f(x))$.
a) Find the general expressions for $g'(x)$ and $h'(x)$.
b) Use the table of values at right to find $g'(2)$ and $h'(4)$.

x	1	2	3	4
$f(x)$	2	3	7	1
$f'(x)$	-2	1	-3	3

a) $g'(x) = \dfrac{1 \cdot f(x) - x \cdot f'(x)}{f(x)^2} = \dfrac{f(x) - x f'(x)}{f(x)^2}$

$g'(2) = \dfrac{f(2) - 2 \cdot f'(2)}{f(2)^2} = \dfrac{3 - 2 \cdot 1}{3^2} = \dfrac{1}{9}$

$h'(x) = f'(f(x)) \cdot f'(x)$

$h'(4) = f'(f(4))\, f'(4)$
$= f'(1) \cdot f'(4) = -2 \cdot 3 = -6$

10. (6pts) An egg thrown upwards from ground height 3 meters has position given by the formula $s(t) = -5t^2 + 8t$.
a) Write the formula for the velocity of the egg at time t.
b) When does the egg reach its maximum height?
c) What is the initial velocity of the egg?

a) $v(t) = s'(t) = -10t + 8$

c) $v(0) = 8$

b) $v(t) = 0$

$-10t + 8 = 0$

$-10t = -8$

$t = \dfrac{8}{10} = \dfrac{4}{5}$ s

11. (10pts) Use implicit differentiation to find y'.

$\sin(x+y) = x^2 + xy + y^2 \quad \Big| \dfrac{d}{dx}$

$\cos(x+y)(1+y') = 2x + 1 \cdot y + xy' + 2yy'$

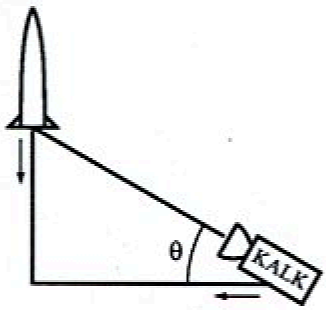
$\cos(x+y) + \cos(x+y) \cdot y' = 2x + y + xy' + 2yy'$

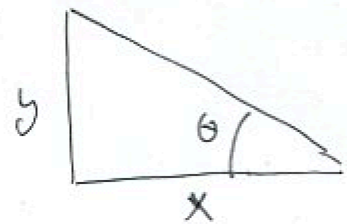
$\cos(x+y) - 2x - y = xy' + 2yy' - \cos(x+y)y'$

$y'(x + 2y - \cos(x+y)) = \cos(x+y) - 2x - y$

$y' = \dfrac{\cos(x+y) - 2x - y}{x + 2y - \cos(x+y)}$

12. (16pts) Late to the event, the crew of TV-station KALK is rushing on a straight road to a vertical landing of a new rocket. Their rooftop camera is aimed at the rocket. KALK's car is approaching the landing site at 35 meters per second and the rocket, located just above the landing site, is descending at 15 meters per second. At what rate is the angle of elevation θ of the line of sight to the rocket changing when the rocket is 3000 meters above ground, and the car is 4000 meters from the landing site? Is the camera tilting lower or higher? *(Hint: sohcahtoa.)*

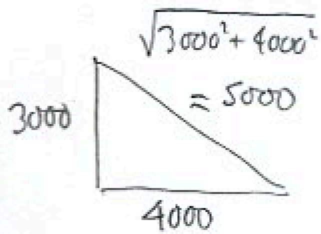




Know: $y' = -15\ m/s$

$x' = -35\ m/s$

$$\boxed{\begin{aligned}\theta' &= \frac{-15\cdot 4000-(-35)\cdot 3000}{5000^2} \\ &= \frac{-60000+105000}{25\,000\,000} = \frac{\cancel{45\,000}^{\,9}}{\cancel{25}_{\,5}\,000\,\cancel{000}} \\ &= \frac{9}{5000} > 0 \text{ so tilting higher}\end{aligned}}$$



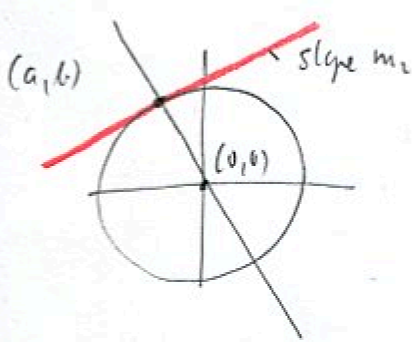
$$\tan\theta = \frac{y}{x} \quad \Big|\ \frac{d}{dt}$$

$$\sec^2\theta\,\theta' = \frac{y'x - yx'}{x^2}$$

$$\theta' = \frac{y'x-yx'}{x^2\sec^2\theta} = \frac{(y'x-yx')\cos^2\theta}{x^2} = \left[\cos\theta = \frac{x}{\sqrt{x^2+y^2}}\right]$$

$$= \frac{y'x-yx'}{\cancel{x^2}}\cdot\frac{\cancel{x^2}}{x^2+y^2} = \frac{y'x-yx'}{x^2+y^2}$$

Bonus. (10pts) It is common knowledge that the tangent line to a circle is perpendicular to the radius at the point of tangency. Show this fact using slopes of the tangent line and the radius line through point (a, b) of the circle $x^2 + y^2 = r^2$. *(Hint: implicit differentiation will make it easier here.)*



slope m_1

$$= \frac{b-0}{a-0} = \frac{b}{a}$$

$$x^2+y^2 = r^2 \quad \Big|\ \frac{d}{dx}$$

$$2x + 2yy' = 0$$

$$2yy' = -2x$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$

$$m_1 \cdot m_2 = \frac{b-0}{a-0}\cdot\left(-\frac{a}{b}\right) = \frac{b}{a}\cdot\left(-\frac{a}{b}\right) = -1$$

so lines are perpendicular