Mathematical Reasoning — Exam 1 **MAT 312, Fall 2017 — D. Ivanšić**	**Name:** ____________________ *Show all your work!*

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) $2 < 5$ or $3 \geq 5$

2. (2pts) If the square of a real number is negative, then that number is negative.

3. (3pts) (universal set=$\mathbf{Z}$) $2x^2 = 5x + 3$

4. (4pts) There exists a $y \in \mathbf{R}$ such that $y^3 + y - 5 = 0$.

5. (3pts) (universal set=$\mathbf{R}$) $3x + 5 < 2x + 4$

Negate the following statements.

6. (3pts) If you know a little bit of English, you can go far.

7. (3pts) Kim Jong Un launches a rocket and takes a bath.

8. (8pts) Use a truth table to prove that $(\neg P \vee Q) \wedge (P \vee \neg Q) \wedge \neg P \equiv \neg P \wedge \neg Q$. (Use however many columns you need.)

P	Q								
T	T								
T	F								
F	T								
F	F								

9. (12pts) Use previously proven logical equivalences to prove the equivalence $(P \implies Q) \implies R \equiv (\neg P \implies R) \wedge (Q \implies R)$. Do not use a truth table.

10. (4pts) Write the converse and contrapositive of the statement: if a real number is greater than 5, then its absolute value is greater than 5.

Converse:

Contrapositive:

11. (8pts) Suppose the following statements are true:
If I ate ice cream, then I ate strawberries.
I did not eat ice cream or I did not eat strawberries.

Determine truth value of the following statement and justify: I ate ice cream.

12. (4pts) Use the roster method to write the set $\{x \in \mathbf{Z} \mid x^2 \text{ is odd and smaller than } 30\}$.

13. (7pts) An integer n is divisible by 7 if $n = 7k$ for some integer k.
a) Write the definition using symbols for quantifiers.
b) Negate the definition using symbols for quantifiers.
c) Finish the sentence: "An integer n is not divisible by 7 if ..."

14. (10pts) There exists an integer n such that for every integer m, if $mn = 24$, then $|n| < 6$.
a) Write this statement using symbols.
b) Write the negation of the statement using symbols.
c) Write the negation of the statement in English.

15. (12pts) Let $\mathbf{Z}$ be the universal set. The following is an open sentence in x:

$$(\exists y \in \mathbf{Z})(x + 2y = 5)$$

a) If $x = 1$, is the statement true?
b) If $x = 6$, is the statement true?
c) Find the truth set (the x's) of the above statement.

16. (15pts) We will call an integer n type-0, type-1, type-2, or type-3 if it can be written in the form $n = 4k$, $n = 4k + 1$, $n = 4k + 2$, or $n = 4k + 3$, respectively, for some integer k. Show that if n is a type-3 integer, then $n^2 - n$ is a type-2 integer. Start with a know-show table if you find it helpful.

Bonus. (10pts) Consider the general quadratic equation $ax^2 + bx + c = 0$. Prove the following statement: if $a > 0$, $b < 0$ and $c > 0$ and the equation has a real solution, then both solutions are positive.

Mathematical Reasoning — Exam 2
MAT 312, Fall 2017 — D. Ivanšić

Name: ____________________________

Show all your work!

1. (14pts) Prove: the sum of squares of three consecutive integers always gives remainder 2 when divided by 3.

2. (14pts) Prove using induction: for every integer $n \geq 0$, $1 + 3 + 3^2 + \cdots + 3^n = \dfrac{3^{n+1} - 2}{2}$.

3. (16pts) Let $a, b \neq 0$ be real numbers. Two of the following statements are true, and one is false. Prove the true ones (one is basic and needs only a little explanation), and justify why the remaining one is false.
a) If a and b are rational, then ab is rational.
b) If a is rational and b is irrational, then ab is irrational.
c) If a and b are irrational, then ab is irrational.

4. (18pts) Consider the statement: for every integer n, n is divisible by 8 if and only if n^2 is divisible by 8.
a) Write the statement as a conjunction of two conditional statements.
b) Determine whether each of the conditional statements is true, and write a proof, if so.
c) Is the original statement true?

5. (14pts) We have shown on homework: for every integer n, if n^2 is even, then n is even. Use this proposition to show directly that $\sqrt{8}$ is irrational, that is, **without** using the fact that $\sqrt{2}$ is irrational.

6. (10pts) Use the triangle inequality to prove that for all real numbers c, d,

$2|c| \leq |c+d| + |c-d|$.

7. (14pts) Prove that for all real numbers x, y, $x^2 + y^2 \geq 6x - 9$.

Bonus. (10pts) Show that the number 345,237,211,897,873,929,146 is not a square of any integer. *Hint: use congruence, but* $\pmod 5$ *and* $\pmod{10}$*, like in our homework problem, will not work.*

Mathematical Reasoning — Exam 3 **MAT 312, Fall 2017 — D. Ivanšić**	**Name:** ________________________ *Show all your work!*

1. (14pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the four sets, two are equal. Use set algebra to show they are equal.

$(B \cap C) - A$ $\qquad$ $(A \cap B) \cap C$ $\qquad$ $(B - A) \cap C$ $\qquad$ $(A - B) \cup (B - C)$

2. (12pts) Let U be the set of integers. Consider the sets $A = \{k \in \mathbf{Z} \mid k \equiv 2 \pmod 4\}$, $B = \{k \in \mathbf{Z} \mid k \text{ is divisible by } 4\}$, $C = \{k \in \mathbf{Z} \mid k < 0\}$ and write the following subsets using the roster method (pattern needs to be obvious).

$A \cap C =$

$B - C =$

$C^c =$

$(A \cup B) \cap C =$

$C - (A \cup B) =$

$B - A =$

3. (12pts) Let $A = \{k \in \mathbf{Z} \mid k \equiv 1 \pmod 3\}$ and $B = \{k \in \mathbf{Z} \mid k \equiv 4 \pmod 6\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $B \subseteq A$? Prove or disprove.

4. (16pts) Let $f : \mathbf{R} \times \mathbf{R} \to [0, \infty)$ be given by $f(x, y) = x^2 + y^2$.
a) Is f surjective? Justify.
b) Is f injective? Justify.
c) Determine the set of preimages of 5. List at least three elements of this set and illustrate it in the plane.

5. (14pts) Let $\mathbf{Z}_4 = \{0, 1, 2, 3\}$, and let $f, g : \mathbf{Z}_4 \to \mathbf{Z}_4$, $f(x) = x^2+4x \pmod 4$, $g(x) = x^2-4 \pmod 4$.
a) Write the table of function values for f and g.
b) The formulas for f and g are different. Are the functions f and g equal?
c) What is the set of preimages of 3 under f?
d) What is the set of preimages of 0 under f?
e) Show that $x^2 + 4x \equiv x^2 - 4 \pmod 4$ for every $x \in \mathbf{Z}_4$. This implies that $f(x) = g(x)$ for every $x \in \mathbf{Z}_4$.

6. (10pts) Let $f(x) = (x - 2)^2 + 7$ and assume the codomain is $\mathbf{R}$.
a) What subset of real numbers is the natural domain for this function?
b) What is the range of this function? Justify your answer.

7. (10pts) Draw arrow diagrams between two copies of **Z** below that illustrate a function $f : \mathbf{Z} \to \mathbf{Z}$ that is:

a) a surjection that is not an injection

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

b) an injection that is not a surjection

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

8. (12pts) Let A, B be subsets of a universal set U. Prove that $A \subseteq B$ if and only if $A \cup B = B$.

Bonus. (10pts) Let $A = \{x \in \mathbf{R} \mid x \neq -1, 1\}$ and let $f : A \to \mathbf{R}$, $f(x) = \dfrac{2}{1 - x^2}$. Determine the range of f.

Mathematical Reasoning — Final Exam
MAT 312, Fall 2017 — D. Ivanšić

Name: ______________________________

Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) $4 > 2$ and $4 \geq -2$.

2. (3pts) For every natural number n, if n is divisible by a natural number greater than n, then n is even.

3. (3pts) (universal set=**Z**) $3x^2 + 14x = 5$.

Negate the following statements. Write English sentences, but you may assist yourself with symbols, if necessary.

4. (3pts) If a real number is less than 1, then its reciprocal is greater than 1.

5. (4pts) For every $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

6. (10pts) Use previously proven logical equivalences to prove the equivalence $P \implies (Q \wedge R) \equiv (P \implies Q) \wedge (P \implies R)$. Do not use a truth table.

7. (16pts) Consider the statement: if x is irrational, then $\sqrt{x}$ is irrational.

a) State the converse and prove or disprove it:

b) State the contrapositive and prove or disprove it:

c) Is this statement true or false (justify): x is irrational if and only if $\sqrt{x}$ is irrational?

8. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\exists y \in \mathbf{R})(x + y^2 = 9)$$

a) If $x = -3$, is the statement true?
b) If $x = 12$, is the statement true?
c) Find the truth set (the x's) of the above statement.

9. (14pts) Prove using induction: for every integer $n \geq 2$, $4^2 + 4^3 + \cdots + 4^n = \dfrac{4^{n+1} - 16}{3}$.

10. (14pts) We have shown on homework: for every integer n, if n^2 is even, then n is even. Use this proposition to show directly that $\sqrt{8}$ is irrational, that is, **without** using the fact that $\sqrt{2}$ is irrational. (Do **not** use the statement "If n^2 is divisible by 8, then n is divisible by 8," because it is not true.)

11. (12pts) Prove that for all real numbers x and y, $x^2 + y^2 + 2 \geq 2y - 2x$.

12. (12pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the three sets, two are equal. Use set algebra to show they are equal.

$(A \cup B) - C$ $\qquad$ $(A - B) \cup (B - C)$ $\qquad$ $(A - C) \cup (B - C)$

13. (12pts) Let $A = \{k \in \mathbf{Z} \mid k \equiv 3 \pmod 5\}$ and $B = \{k \in \mathbf{Z} \mid k^2 - k \equiv 1 \pmod 5\}$.
a) Prove $A \subseteq B$.
b) Prove $B \subseteq A$ by proving the equivalent: $A^c \subseteq B^c$.

14. (16pts) Let $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be given by $f(x, y) = y + x^2$.
a) Is f surjective? Justify.
b) Is f injective? Justify.
c) Determine the set of preimages of 5. List at least three elements of this set and illustrate it in the plane.

15. (5pts) Draw an arrow diagram between the provided two copies of $\mathbf{Z}$ that illustrates a function $f : \mathbf{Z} \to \mathbf{Z}$ that is an injection and is not a surjection (pattern needs to be obvious).

$$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$$

$$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$$

16. (12pts) Let A, B be subsets of a universal set U. Prove that $A \subseteq B$ if and only if $A \cup B = B$.

Bonus. (10pts) Consider the general quadratic equation $ax^2 + bx + c = 0$. Prove the following statement: if $a > 0$, $b < 0$ and $c > 0$ and the equation has a real solution, then both solutions are positive.