Mathematical Reasoning — Final Exam
MAT 312, Fall 2015 — D. Ivanšić

Name: ______________________________

Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (3pts) If an integer is greater than 3, then it is less than 101.

2. (3pts) For every x, if $x^2 < -1$, then $3x - 1 = 4x$.

3. (3pts) (universal set=$\mathbf{R}$) $\sqrt{x^2} = x$.

Negate the following statements. Write English sentences, but you may assist yourself with symbols, if necessary.

4. (3pts) If n is an integer divisible by 4, then its sum of digits is divisible by 4.

5. (4pts) For every natural number m, there exists a natural number n such that $m - n$ is a square of a natural number.

6. (4pts) Write the converse and contrapositive of the statement: if n is divisible by 7, then n^2 is divisible by 7.

Converse:

Contrapositive:

7. (10pts) Use previously proven logical equivalences to prove the equivalence $P \implies (Q \implies R) \equiv (P \wedge Q) \implies R$. Do not use a truth table.

8. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\exists y \in \mathbf{R})(3x + y^2 = 4)$$

a) If $x = -1$, is the statement true?
b) If $x = 3$, is the statement true?
c) Find the truth set (the x's) of the above statement.

9. (10pts) Let $p \neq 0$ be a rational number. Prove: for every real number x, if x is irrational, then $\dfrac{p}{1+x}$ is irrational.

10. (14pts) Prove using induction: for every $n \in \mathbf{N}$, $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n} = \frac{1}{2}\left(3 - \frac{1}{3^n}\right)$.

11. (14pts) Consider the statement: for every integer n, n is divisible by 4 if and only if $2n^2 + 5n$ is divisible by 4.
a) Write the statement as a conjunction of two conditional statements.
b) Determine whether each of the conditional statements is true, and write a proof, if so.
c) Is the original statement true?

12. (10pts) Prove that for every real number a, if $a > -3$, then $a + 1 + \dfrac{1}{a+3} \geq 0$.

13. (12pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the three sets, two are equal. Use set algebra to show they are equal.

$(A \cap B) - (A \cap C)$ $\qquad\qquad$ $(A - C) \cup (B - C)$ $\qquad\qquad$ $(A \cap B) - C$

14. (12pts) Let $A = \{k \in \mathbf{Z} \mid k \equiv 5 \pmod 6\}$ and $B = \{k \in \mathbf{Z} \mid k \equiv 2 \pmod 3\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $B \subseteq A$? Prove or disprove.

15. (14pts) Let $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ be given by $f(m, n) = 2m - 3n$.
a) Evaluate $f(0, 7)$ and $f(1, -3)$.
b) Determine the set of preimages of 1. List at least three elements of this set and illustrate it in the plane.
c) Is this function injective?
d) Is this function surjective? *(Hint: if $2m - 3n = 1$ has a solution, it's easy to find the solution of $2m - 3n = k$. How?)*

16. (12pts) Let A, B be subsets of a universal set U. Prove that $A \subseteq B$ if and only if $A \cap B^c = \emptyset$.

17. (10pts) Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^2 - 5x + 3$. Determine the range of this function algebraically.

Bonus 1. (8pts) Show by example that quantifiers are not "commutative." That is, give an example of an open sentence $P(x, y)$ about real numbers x and y, so that the statements below have opposite truth values (justify why they do).

$(\forall x \in \mathbf{R})(\exists y \in \mathbf{R})P(x, y)$ $\qquad\qquad$ $(\exists y \in \mathbf{R})(\forall x \in \mathbf{R})P(x, y)$

Bonus 2. (7pts) If x and y are irrational and $y \neq \dfrac{1}{x}$, does it follow that xy is irrational? *(Hint: difference of squares.)*