Mathematical Reasoning — Final Exam MAT 312, Fall 2017 — D. Ivanšić	Name: Saul Ocean

Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) $\underbrace{4 > 2}_{\text{true}}$ and $\underbrace{4 \geq -2}_{\text{true}}$.

$T \wedge T \equiv T$ so true.

2. (3pts) For every ~~integer~~ natural number n, if n is divisible by an integer ~~greater~~ natural number than n, then n is even.

True, because "F ⇒ anything" is true, and no natural number is divisible by a greater natural number.

3. (3pts) (universal set=**Z**) $3x^2 + 14x = 5$. open sentence $\{-5\}$ is the truth set

$3x^2+14x-5=0$

$x = \frac{-14 \pm \sqrt{14^2+4 \cdot 5 \cdot 3}}{2 \cdot 3} = \frac{-14 \pm \sqrt{156+60}}{6} = \frac{-14 \pm \sqrt{256}}{6} = \frac{-14 \pm 16}{6} = -5, \frac{1}{3}$ not in $\mathbb{Z}$

Negate the following statements. Write English sentences, but you may assist yourself with symbols, if necessary.

4. (3pts) If a real number is less than 1, then its reciprocal is greater than 1.

$\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$ A real number is less than one and its reciprocal is less than or equal to 1.

5. (4pts) For every $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

$(\forall y \in B)(\exists x \in A)(f(x) = y) \xrightarrow{\text{negate}} (\exists y \in B)(\forall x \in A)(f(x) \neq y)$

There exists a $y \in B$ such that for every $x \in A$, $f(x) \neq y$.

6. (10pts) Use previously proven logical equivalences to prove the equivalence $P \Longrightarrow (Q \wedge R) \equiv (P \Longrightarrow Q) \wedge (P \Longrightarrow R)$. Do not use a truth table.

$P \Rightarrow (Q \wedge R) \equiv \neg P \vee (Q \wedge R) \equiv (\neg P \vee Q) \wedge (\neg P \vee R) \equiv (P \Rightarrow Q) \wedge (P \Rightarrow R)$

7. (16pts) Consider the statement: if x is irrational, then $\sqrt{x}$ is irrational.

a) State the converse and prove or disprove it: If $\sqrt{x}$ is irrational, then x is irrational.

False: $\sqrt{2}$ is irrational, but 2 is not irrational.

b) State the contrapositive and prove or disprove it: If $\sqrt{x}$ is rational, then x is rational.

True. Suppose $\sqrt{x} = g$ for some $g \in \mathbb{Q}$. Then $\sqrt{x}^2 = g^2$, $x = g^2$,

which is rational, due to closure of $\mathbb{Q}$ under multiplication,

Thus x is rational.

c) Is this statement true or false (justify): x is irrational if and only if $\sqrt{x}$ is irrational?

If the original statement is denoted $P \Rightarrow Q$, this statement is $P \Leftrightarrow Q$.

In a) we showed that $Q \Rightarrow P$ is false. Thus, the biconditional

statement $P \Leftrightarrow Q$ is false, too.

8. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\exists y \in \mathbf{R})(x + y^2 = 9)$$

a) If $x = -3$, is the statement true?
b) If $x = 12$, is the statement true?
c) Find the truth set (the x's) of the above statement.

a) $(\exists y \in \mathbb{R})(-3 + y^2 = 9)$

$\downarrow$

$y^2 = 12$

$y = \pm\sqrt{12}$ a real number

The eq. $3 + y^2 = 9$ has a solution,

so statement is true for $x = -3$

b) $(\exists y \in \mathbb{R})(12 + y^2 = 9)$

$y^2 = -3$

no real solution, since $y^2 \geq 0$ for all $y \in \mathbb{R}$

Statement is false for $x = 12$.

c) For which x does $x + y^2 = 9$ have a real solution in y?

$y^2 = 9 - x$, $y = \pm\sqrt{9 - x}$

This has a solution iff $9 - x \geq 0$

so $x \leq 9$.

Truth set is $(-\infty, 9]$

9. (14pts) Prove using induction: for every integer $n \geq 2$, $4^2 + 4^3 + \cdots + 4^n = \dfrac{4^{n+1} - 16}{3}$.

Basis: $n=2$ $\quad 4^2 \overset{?}{=} \dfrac{4^3-16}{3} = \dfrac{48}{3}$

$4^2 \overset{?}{=} 16$ is true

Induction step: Suppose it is true for $n=k$, so

$$4^2+4^3+\dots+4^k = \frac{4^{k+1}-16}{3} \quad \Big| + 4^{k+1}$$

$$4^2+4^3+\dots+4^{k+1} = \frac{4^{k+1}-16}{3} + 4^{k+1}$$

$$= \frac{4^{k+1}-16+3\cdot 4^{k+1}}{3}$$

$$= \frac{4\cdot 4^{k+1}-16}{3}$$

$$4^2+4^3+\dots+4^{k+1} = \frac{4^{k+2}-16}{3},$$ which is exactly the statement for $n=k+1$

10. (14pts) We have shown on homework: for every integer n, if n^2 is even, then n is even. Use this proposition to show directly that $\sqrt{8}$ is irrational, that is, **without** using the fact that $\sqrt{2}$ is irrational. (Do **not** use the statement "If n^2 is divisible by 8, then n is divisible by 8," because it is not true.)

Suppose $\sqrt{8} = \frac{m}{n}$, where m, n are integers and $\frac{m}{n}$ is reduced. Then $8 = \frac{m^2}{n^2}$, $m^2 = 8n^2 = 2(4n^2)$. Thus m^2 is even, so m is even and can be written as $m = 2k$. This gives $(2k)^2 = 8n^2$, $4k^2 = 8n^2$ so $k^2 = 2n^2$, making k^2 even, so k is even, $k = 2l$ for some integer l. Then $4(2l)^2 = 8n^2$, $16l^2 = 8n^2$, so $n^2 = 2l^2$ implying that n^2 is even so n is even. We get that both m and n are even, contradicting the assumption that $\frac{m}{n}$ is reduced. Thus, $\sqrt{8}$ is irrational.

11. (12pts) Prove that for all real numbers x and y, $x^2 + y^2 + 2 \geq 2y - 2x$.

Investigation.

$x^2+y^2+2 \geq 2y-2x$

$x^2+2x+y^2-2y+2 \geq 0$

$x^2+2x+1+y^2-2y+1 \geq 0$

$(x+1)^2+(y-1)^2 \geq 0$

which is true.

Proof. For any $x, y \in \mathbb{R}$,

$(x+1)^2 \geq 0$

$(y-1)^2 \geq 0$

Adding the inequalities, we get

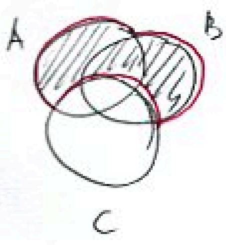
$(x+1)^2+(y-1)^2 \geq 0$

$x^2+2x+1+y^2-2y+1 \geq 0$

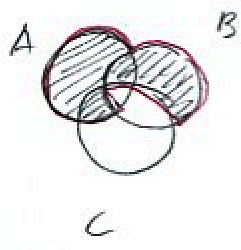
$x^2+y^2+2 \geq 2y-2x$.

12. (12pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the three sets, two are equal. Use set algebra to show they are equal.

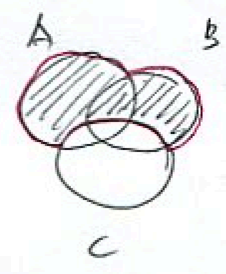
$(A \cup B) - C$



$(A - B) \cup (B - C)$



$(A - C) \cup (B - C)$



It appears that $(A \cup B) - C = (A-C) \cup (B-C)$

$A \cup B - C = (A \cup B) \cap C^c = (A \cap C^c) \cup (B \cap C^c) = (A-C) \cup (B-C)$

13. (12pts) Let $A = \{k \in \mathbf{Z} \mid k \equiv 3 \pmod 5\}$ and $B = \{k \in \mathbf{Z} \mid k^2 - k \equiv 1 \pmod 5\}$.
a) Prove $A \subseteq B$.
b) Prove $B \subseteq A$ by proving the equivalent: $A^c \subseteq B^c$.

Consider the congruence chart:

$k \equiv \square$	k^2-k	$k^2-k \equiv \square \bmod 5$
0	0	0
1	0	0
2	2	2
3	6	1
4	12	2

a) $A \subseteq B$,

Let $k \in A$. Then $k \equiv 3 \pmod 5$, giving $k^2 - k \equiv 1 \pmod 5$ from chart, so $k \in B$.

b) Let $k \in A^c$. Then $k \not\equiv 3 \pmod 5$, so $k \equiv 0,1,2,4 \pmod 5$ giving $k^2 - k \equiv 0, 2 \pmod 5$ by the chart, so $k^2 - k \not\equiv 1 \pmod 5$ and $k \in B^c$.

14. (16pts) Let $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be given by $f(x, y) = y + x^2$.
a) Is f surjective? Justify.
b) Is f injective? Justify.
c) Determine the set of preimages of 5. List at least three elements of this set and illustrate it in the plane.

a) Can we solve $f(x,y) = z$ for every $z \in \mathbb{R}$?

$$y + x^2 = z$$

has a solution: $y = z$, $x = 0$

Thus, for every $z \in \mathbb{R}$, $f(0, z) = z$, so f is surjective.

b) No. For example, $(0,0) \neq (1,-1)$

But $f(0,0) = 0 = f(1,-1)$

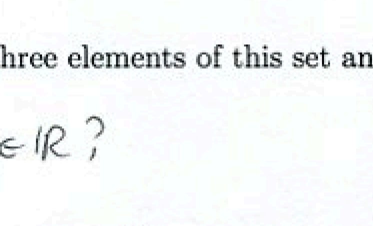
c) $f(x,y) = 5$

$$y + x^2 = 5$$

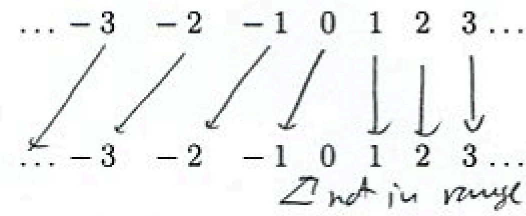
Some elements of set of preimages are:

$(0,5), (2,1), (-2,1)$

$y = 5 - x^2$ is a parabola in the plane



15. (5pts) Draw an arrow diagram between the provided two copies of **Z** that illustrates a function $f : \mathbf{Z} \to \mathbf{Z}$ that is an injection and is not a surjection (pattern needs to be obvious).



16. (12pts) Let A, B be subsets of a universal set U. Prove that $A \subseteq B$ if and only if $A \cup B = B$.

We prove;

$\Rightarrow$) If $A \subseteq B$, then $A \cup B = B$. Assume $A \subseteq B$.

a) $A \cup B \subseteq B$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \in B$, since $A \subseteq B$. If $x \in B$, then $x \in B$ obviously. In either case we arrive at $x \in B$.

b) $B \subseteq A \cup B$. Let $x \in B$. Then $x \in B$ or $x \in A$ is true, so $x \in A \cup B$.

$\Leftarrow$) If $A \cup B = B$, then $A \subseteq B$. Assume $A \cup B = B$.

Let $x \in A$. Then $x \in A$ or $x \in B$, so $x \in A \cup B$. Since $A \cup B = B$, we conclude $x \in B$.

Bonus. (10pts) Consider the general quadratic equation $ax^2 + bx + c = 0$. Prove the following statement: if $a > 0$, $b < 0$ and $c > 0$ and the equation has a real solution, then both solutions are positive.

The solutions to $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Assume the equation has a real solution and that $a > 0$, $b < 0$ and $c > 0$. Then $b^2 - 4ac \geq 0$ and $4ac > 0$, so $b^2 - 4ac < b^2$. We get

$0 \leq b^2 - 4ac < b^2$ Take $\sqrt{\ }$ (all are ≥ 0)

$0 \leq \sqrt{b^2 - 4ac} < \sqrt{b^2}$ $\sqrt{b^2} = |b| = -b$, since $b < 0$

$0 \leq \sqrt{b^2 - 4ac} < -b$ $| -\sqrt{b^2 - 4ac}$

$0 < -b - \sqrt{b^2 - 4ac}$ $| \div 2a$, $2a > 0$

$0 < \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$, so one of the solutions is positive

For the other solution, $-b > 0$ and $\sqrt{b^2 - 4ac} \geq 0$

so $-b + \sqrt{b^2 - 4ac} > 0$

$\dfrac{-b + \sqrt{b^2 - 4ac}}{2a} > 0.$