Mathematical Reasoning — Exam 3
MAT 312, Fall 2017 — D. Ivanšić

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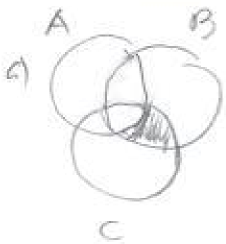
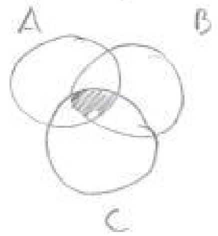
Show all your work!

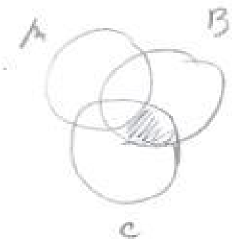
1. (14pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the four sets, two are equal. Use set algebra to show they are equal.

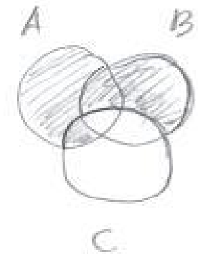
$(B \cap C) - A$ $\qquad$ $(A \cap B) \cap C$ $\qquad$ $(B - A) \cap C$ $\qquad$ $(A - B) \cup (B - C)$

a)









b) It appears that $(B \cap C) - A = (B - A) \cap C$

$(B \cap C) - A = (B \cap C) \cap A^c = (B \cap A^c) \cap C = (B - A) \cap C$

2. (12pts) Let U be the set of integers. Consider the sets $A = \{k \in \mathbf{Z} \mid k \equiv 2 \pmod 4\}$, $B = \{k \in \mathbf{Z} \mid k \text{ is divisible by } 4\}$, $C = \{k \in \mathbf{Z} \mid k < 0\}$ and write the following subsets using the roster method (pattern needs to be obvious).

$A \cap C = \{\ldots, -14, -10, -6, -2\}$

$B - C = \{0, 4, 8, 12, \ldots\}$

$C^c = \{0, 1, 2, 3, 4, \ldots\}$

$(A \cup B) \cap C =$ (all even) $\cap C = \{\ldots, -8, -6, -4, -2\}$

$C - (A \cup B) = C -$ (all even) $= \{\ldots, -9, -7, -5, -3, -1\}$

$B - A = B = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

$A = \{4k+2 \mid k \in \mathbb{Z}\} = \{\ldots -10, -6, -2, 2, 6, 10, \ldots\}$ $\quad$ $B = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

$C = \{\ldots, -4, -3, -2, -1\}$

3. (12pts) Let $A = \{k \in \mathbf{Z} \mid k \equiv 1 \pmod 3\}$ and $B = \{k \in \mathbf{Z} \mid k \equiv 4 \pmod 6\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $B \subseteq A$? Prove or disprove.

$A = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$

$B = \{\ldots, -8, -2, 4, 10, 16, \ldots\}$

a) $A \nsubseteq B$ because $1 \in A$ but $1 \notin B$

b) Yes. Let $k \in B$. Then $k \equiv 4 \pmod 6$, so $k = 6q + 4$ for some $q \in \mathbb{R}$

Then $k = 3 \cdot 2q + 3 + 1 = 3(2q+1)+1$, so $k - 1 = 3(2q+1)$, which

means $3 \mid k-1$, so $k \equiv 1 \pmod 3$, i.e. $k \in A$.

4. (16pts) Let $f : \mathbf{R} \times \mathbf{R} \to [0, \infty)$ be given by $f(x, y) = x^2 + y^2$.
a) Is f surjective? Justify.
b) Is f injective? Justify.
c) Determine the set of preimages of 5. List at least three elements of this set and illustrate it in the plane.

a) f is surjective: given $z \in [0, \infty)$, $z \geq 0$.

Try to find a solution for $x^2 + y^2 = z$

A possible solution is $x = 0$, $y = \sqrt{z}$, so $f(0, \sqrt{z}) = z$

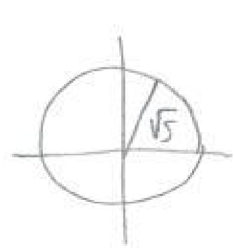
b) f is not injective. For example, $(0,2) \neq (2,0)$, but $f(0,2) = 4 = f(2,0)$

c) $f(x,y) = 5$

$x^2 + y^2 = 5$

$x^2 + y^2 = 5$ is a circle in plane,
radius $\sqrt{5}$,
center $(0,0)$

$\left.\begin{array}{l} (\sqrt{5}, 0) \\ (1, 2) \\ (-2, -1) \end{array}\right\}$ are all preimages of 5



5. (14pts) Let $\mathbf{Z}_4 = \{0, 1, 2, 3\}$, and let $f, g : \mathbf{Z}_4 \to \mathbf{Z}_4$, $f(x) = x^2+4x \pmod 4$, $g(x) = x^2-4 \pmod 4$.
a) Write the table of function values for f and g.
b) The formulas for f and g are different. Are the functions f and g equal?
c) What is the set of preimages of 3 under f?
d) What is the set of preimages of 0 under f?
e) Show that $x^2 + 4x \equiv x^2 - 4 \pmod 4$ for every $x \in \mathbf{Z}_4$. This implies that $f(x) = g(x)$ for every $x \in \mathbf{Z}_4$.

a)

x	x^2+4x	x^2-4	$f(x)$	$g(x)$
0	0	0	0	0
1	5	-3	1	1
2	12	0	0	0
3	21	5	1	1

c) $f(x)=3$ no such x

set of preimages of $3 = \emptyset$

d) $f(x)=0$ $x = 0, 2$

set of preimages of $0 = \{0, 2\}$

b) since $f(x) = g(x)$ for every $x \in \mathbb{Z}_4$ and domains and codomains of f, g are same, functions are equal.

6. (10pts) Let $f(x) = (x-2)^2 + 7$ and assume the codomain is $\mathbf{R}$.
a) What subset of real numbers is the natural domain for this function?
b) What is the range of this function? Justify your answer.

a) all real numbers

b) For which y does $(x-2)^2+7 = y$ have a solution?

Range $f = [7, \infty)$

$(x-2)^2 = y-7$ has sol. only if $y \geq 7$ since $(x-2)^2 \geq 0$

$x-2 = \pm\sqrt{y-7}$

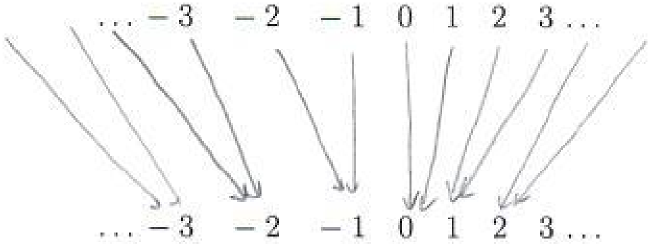
$x = 2 \pm \sqrt{y-7}$

Has solution if and only if $y-7 \geq 0$

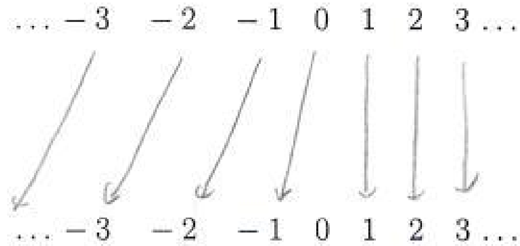
i.e. $y \geq 7$

7. (10pts) Draw arrow diagrams between two copies of $\mathbf{Z}$ below that illustrate a function $f : \mathbf{Z} \to \mathbf{Z}$ that is:

a) a surjection that is not an injection



b) an injection that is not a surjection



8. (12pts) Let A, B be subsets of a universal set U. Prove that $A \subseteq B$ if and only if $A \cup B = B$.

$\Rightarrow$) If $A \subseteq B$, then $A \cup B = B$

We show: $A \cup B \subseteq B$: Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \in B$, since $A \subseteq B$. If $x \in B$, then $x \in B$. In either case, we get $x \in B$, so $A \cup B \subseteq B$

$B \subseteq A \cup B$: Let $x \in B$. Then $x \in B$ or $x \in A$ is true, so $x \in A \cup B$

$\Leftarrow$) If $A \cup B = B$, then $A \subseteq B$.

Let $x \in A$. Then $x \in A \cup B$, but this implies $x \in B$, since $A \cup B = B$.

Thus, $A \subseteq B$.

Bonus. (10pts) Let $A = \{x \in \mathbf{R} \mid x \neq -1, 1\}$ and let $f : A \to \mathbf{R}$, $f(x) = \dfrac{2}{1-x^2}$. Determine the range of f.

For which y does $\dfrac{2}{1-x^2} = y$ have a sol. for x

$$\frac{1-x^2}{2} = \frac{1}{y}$$

$$1 - x^2 = \frac{2}{y}$$

$$1 - \frac{2}{y} = x^2$$

$$x = \pm\sqrt{1 - \frac{2}{y}}$$

For this to be defined we must have

$$1 - \frac{2}{y} \geq 0 \text{ and } y \neq 0 \quad | \cdot y^2$$

$$y^2 - 2y \geq 0$$

$$y(y-2) \geq 0$$

sol to $y^2 - 2y \geq 0$, $y \neq 0$

Thus, range is

$$(-\infty, 0) \cup [2, \infty)$$