Mathematical Reasoning — Exam 2 MAT 312, Fall 2017 — D. Ivanšić

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Show all your work!

1. (14pts) Prove: the sum of squares of three consecutive integers always gives remainder 2 when divided by 3.

Three consecutive integers can be written as $k^2+(k+1)^2+(k+2)^2$ for some $k \in \mathbb{Z}$. Then

$$k^2+(k+1)^2+(k+2)^2 = k^2+k^2+2k+1+k^2+4k+4$$
$$= 3k^2+6k+5$$
$$= 3k^2+6k+3+2$$
$$= 3(k^2+2k+1)+2$$

Hence the remainder of dividing $k^2+(k+1)^2+(k+2)^2$ by 3 is 2.

2. (14pts) Prove using induction: for every integer $n \geq 0$, $1+3+3^2+\cdots+3^n = \dfrac{3^{n+1}-\cancel{2}\,1}{2}$.

Basis step: Let $n=0$ $\quad 1 = \dfrac{3^{0+1}-1}{2}$

$1 = \dfrac{2}{2}$ yes

Induction step: Suppose statement is true for $n=k$:

$$1+3+3^2+\cdots+3^k = \frac{3^{k+1}-1}{2} \quad \Big) + 3^{k+1}$$

$$1+3+3^2+\cdots+3^{k+1} = \frac{3^{k+1}-1}{2}+3^{k+1} = \frac{3^{k+1}-1+2\cdot 3^{k+1}}{2}$$

$$= \frac{3\cdot 3^{k+1}-1}{2} = \frac{3^{k+2}-1}{2}$$

which gives $1+3+3^2+\cdots+3^{k+1} = \dfrac{3^{k+2}-1}{2}$, the statement for $n=k+1$

3. (16pts) Let $a, b \neq 0$ be real numbers. Two of the following statements are true, and one is false. Prove the true ones (one is basic and needs only a little explanation), and justify why the remaining one is false.
a) If a and b are rational, then ab is rational.
b) If a is rational and b is irrational, then ab is irrational.
c) If a and b are irrational, then ab is irrational.

a) True, due to closure of rational numbers under multiplication.

b) True. Prove by contradiction: suppose $a, b \neq 0$ and a is rational, b irrational and that ab is rational. Then $ab = g$ for some $g \in \mathbb{Q}$.
Then $b = \frac{g}{a}$, a rational number due to closure of division in $\mathbb{Q}$
contradicting that b is irrational.

c) False: $\sqrt{2} \cdot \sqrt{2} = 2$
irrational, rational

4. (18pts) Consider the statement: for every integer n, n is divisible by 8 if and only if n^2 is divisible by 8.
a) Write the statement as a conjunction of two conditional statements.
b) Determine whether each of the conditional statements is true, and write a proof, if so.
c) Is the original statement true?

a) 1) If n is divisible by 8, then n^2 is divisible by 8 <u>and</u>
2) If n^2 is divisible by 8, then n is divisible by 8.

b) Consider the congruence table mod 8

$n \equiv \square$	$n^2 \equiv \square$ (mod 8)
0	0
1	1
2	4
3	1
4	0
5	1
6	4
7	1

From table, we see:
1) If $n \equiv 0 \pmod 8$ then $n^2 \equiv 0 \pmod 8$, proving $(\Rightarrow)$
2) It is possible to have $n^2 \equiv 0 \pmod 8$ and $n \not\equiv 0 \bmod 8$
Example: $n = 4$, $8 \mid 4^2$, yet $8 \nmid 4$

c) Since one of the conditional statements is false, the biconditional statement is false.

5. (14pts) We have shown on homework: for every integer n, if n^2 is even, then n is even. Use this proposition to show directly that $\sqrt{8}$ is irrational, that is, **without** using the fact that $\sqrt{2}$ is irrational.

Prove by contradiction. Suppose $\sqrt{8}$ is rational, that is $\sqrt{8} = \frac{m}{n}$ for some integers m, n, where $\frac{m}{n}$ is reduced.

$\sqrt{8} = \frac{m}{n}$

$8 = \frac{m^2}{n^2}$

$8n^2 = m^2$

$m^2 = 2 \cdot 4n^2$, so m^2 is even, which implies m is even, $m = 2k$ for some integer k.

$8n^2 = (2k)^2$

$8n^2 = 4k^2$

$2n^2 = k^2$, so k is even, $k = 2l$ for some integer l.

Then $2n^2 = 4l^2$

$n^2 = 2l^2$, so n^2 is even, which implies n is even,

We get that both m and n are even, contradicting the assumption that $\frac{m}{n}$ is reduced.

6. (10pts) Use the triangle inequality to prove that for all real numbers c, d,

$2|c| \leq |c+d| + |c-d|$.

$|c+d+(c-d)| \overset{\text{by triangle inequality}}{\leq} |c+d| + |c-d|$

$|2c| \leq |c+d| + |c-d|$

$2|c| \leq |c+d| + |c-d|$

7. (14pts) Prove that for all real numbers x, y, $x^2 + y^2 \geq 6x - 9$.

Investigation:

$x^2+y^2 \geq 6x-9$

$x^2-6x+9+y^2 \geq 0$

$(x-3)^2+y^2 \geq 0$

True, since both are positive.

Proof:

Let x, y be real numbers.

Then $(x-3)^2 \geq 0$ and $y^2 \geq 0$.

Adding the inequalities, we get

$(x-3)^2+y^2 \geq 0$

$x^2-6x+9+y^2 \geq 0$

$x^2+y^2 \geq 6x-9$

Bonus. (10pts) Show that the number 345,237,211,897,873,929,146 is not a square of any integer. *Hint: use congruence, but* (mod 5) *and* (mod 10), *like in our homework problem, will not work.*

Let m be the number shown, and suppose $m = n^2$ for some integer n. Consider congruence (mod 4)

$n \equiv \square$	$n^2 \equiv \square \pmod 4$
0	0
1	1
2	0
3	1

This shows that $n^2 \equiv 0, 1 \pmod 4$

so $m \equiv 0, 1 \pmod 4$.

This contradicts the fact that

$m = 100g + 46$

$= 4(25g + 11) + 2$

so $m \equiv 2 \pmod 4$