Mathematical Reasoning — Exam 1
MAT 312, Fall 2017 — D. Ivanšić

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Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) $2 < 5$ or $3 \geq 5$

true since $2<5$ is true

2. (2pts) If the square of a real number is negative, then that number is negative.

True because the hypothesis (square of a real number is negative) is false. $F \Rightarrow Q$ is always true.

3. (3pts) (universal set=$\mathbf{Z}$) $2x^2 = 5x + 3$ open sentence, truth set: $\{3\}$

$2x^2-5x-3=0$

$x=\frac{-(-5)\pm\sqrt{(-5)^2-4\cdot 2(-3)}}{2\cdot 2}=\frac{5\pm\sqrt{49}}{4}=\frac{5\pm 7}{4}=3,-\frac{1}{2}$ ← not in $\mathbb{Z}$

4. (4pts) There exists a $y \in \mathbf{R}$ such that $y^3 + y - 5 = 0$.

True, because the graph of a cubic polynomial, N or Ч always crosses the x-axis,

5. (3pts) (universal set=$\mathbf{R}$) $3x + 5 < 2x + 4$ open sentence

$3x-2x<4-5$ $(-\infty,-1)$

$x<-1$

Negate the following statements.

6. (3pts) If you know a little bit of English, you can go far.

you know a little bit of English and you cannot go far,

7. (3pts) Kim Jong Un launches a rocket and takes a bath.

Kim Jong Un does not launch a rocket or does not take a bath.

8. (8pts) Use a truth table to prove that $(\neg P \vee Q) \wedge (P \vee \neg Q) \wedge \neg P \equiv \neg P \wedge \neg Q$. (Use however many columns you need.)

P	Q	$\neg P$	$\neg Q$	$\neg P \vee Q$	$P \vee \neg Q$	$(\neg P \vee Q) \wedge (P \vee \neg Q) \wedge \neg P$	$\neg P \wedge \neg Q$			
T	T	F	F	T	T	F	F			
T	F	F	T	F	T	F	F			
F	T	T	F	T	F	F	F			
F	F	T	T	T	T	T	T			

are same

9. (12pts) Use previously proven logical equivalences to prove the equivalence $(P \Longrightarrow Q) \Longrightarrow R \equiv (\neg P \Longrightarrow R) \wedge (Q \Longrightarrow R)$. Do not use a truth table.

$$(P \Rightarrow Q) \Rightarrow R \equiv \neg(P \Rightarrow Q) \vee R$$
$$\equiv (P \wedge \neg Q) \vee R \quad \text{distribute}$$
$$\equiv (P \vee R) \wedge (\neg Q \vee R)$$
$$\equiv (\neg P \Rightarrow R) \wedge (Q \Rightarrow R)$$

10. (4pts) Write the converse and contrapositive of the statement: if a real number is greater than 5, then its absolute value is greater than 5.

Converse: If the absolute value of a real number is greater than 5, the number is greater than 5

Contrapositive: If the absolute value of a number is less than or equal to 5 then the real number is less than or equal to 5.

11. (8pts) Suppose the following statements are true:

$P \Rightarrow Q$ If I ate ice cream, then I ate strawberries.

$\neg P \vee \neg Q$ I did not eat ice cream or I did not eat strawberries.

P = I ate ice cream
Q = I ate strawberries

Determine truth value of the following statement and justify: I ate ice cream.

P	Q	$P \Rightarrow Q$	$\neg P \vee \neg Q$
T	T	T	F
T	F	F	T
F	T	T	T
F	F	T	T

The rows marked are when both $P \Rightarrow Q$ and $\neg P \vee \neg Q$ are true. In both cases P is false, so "I ate ice cream" is false

OR

Suppose "I ate ice cream" is true. Then "I ate strawberries" is true, But then "I did not eat ice cream or I did not eat strawberries" is false, which can't be. Therefore "I ate ice cream" is false.

12. (4pts) Use the roster method to write the set $\{x \in \mathbf{Z} \mid x^2 \text{ is odd and smaller than } 30\}$.

$\{1, -1, 3, -3, 5, -5\}$

13. (7pts) An integer n is divisible by 7 if $n = 7k$ for some integer k.
a) Write the definition using symbols for quantifiers.
b) Negate the definition using symbols for quantifiers.
c) Finish the sentence: "An integer n is not divisible by 7 if ..."

a) n is divisible by 7 if $(\exists k \in \mathbb{Z})(n = 7k)$

b) n is not divisible by 7 if $(\forall k \in \mathbb{Z})(n \neq 7k)$

c) An integer n is not divisible by 7 if for every integer k, $n \neq 7k$.

14. (10pts) There exists an integer n such that for every integer m, if $mn = 24$, then $|n| < 6$.
a) Write this statement using symbols.
b) Write the negation of the statement using symbols.
c) Write the negation of the statement in English.

a) $(\exists n \in \mathbb{Z})(\forall m \in \mathbb{Z})(mn = 24 \Rightarrow |n| < 6)$

b) $(\forall n \in \mathbb{Z})(\exists m \in \mathbb{Z})(mn = 24 \text{ and } |n| \geq 6)$

c) For every integer n, there exists an integer m such that $mn = 24$ and $|n| \geq 6$.

15. (12pts) Let $\mathbf{Z}$ be the universal set. The following is an open sentence in x:

$$(\exists y \in \mathbf{Z})(x + 2y = 5)$$

a) If $x = 1$, is the statement true?
b) If $x = 6$, is the statement true?
c) Find the truth set (the x's) of the above statement.

a) $x = 1$ $\quad 1 + 2y = 5$, $2y = 4$, $y = 2$ $\quad$ True, since equation has a solution

b) $x = 6$ $\quad 6 + 2y = 5$, $2y = -1$, $y = -\frac{1}{2}$ $\quad$ False, since equation does not have an integer solution

c) $x + 2y = 5$ has an integer solution for y if $y = \frac{5-x}{2}$ is an integer,
That is $5 - x$ is even
$5 - x = 2k$
$x = 2k + 5 = 2(k+2) + 1$, x is odd.
Truth set: $\boxed{\text{odd integers}}$

16. (15pts) We will call an integer n type-0, type-1, type-2, or type-3 if it can be written in the form $n = 4k$, $n = 4k+1$, $n = 4k+2$, or $n = 4k+3$, respectively, for some integer k. Show that if n is a type-3 integer, then $n^2 - n$ is a type-2 integer. Start with a know-show table if you find it helpful.

Suppose n is a type-3 integer. Then there exists an integer k s.t. $n = 4k+3$. Then:

$$n^2 - n = (4k+3)^2 - (4k+3) = 16k^2 + 24k + 9 - 4k - 3$$
$$= 16k^2 + 20k + 6 \quad 4+2$$
$$= 4(4k^2 + 5k + 1) + 2$$
$$= 4g + 2 \quad \text{for } g = 4k^2 + 5k + 1$$

Since $n^2 - n$ can be written in form $4g+2$, it is a type-2 integer.

Bonus. (10pts) Consider the general quadratic equation $ax^2 + bx + c = 0$. Prove the following statement: if $a > 0$, $b < 0$ and $c > 0$ and the equation has a real solution, then both solutions are positive.

Solutions are $x = \dfrac{-b \pm \sqrt{b^2 - ac}}{2a}$. Suppose $a > 0$, $b < 0$ and $c > 0$ and the equation has a solution. Then $b^2 - 4ac \geq 0$. Since $-b > 0$, the solution $\dfrac{-b + \sqrt{b^2 - ac}}{2a}$ is positive $\left(\dfrac{\text{pos} + \text{pos}}{\text{pos}}\right)$.

Because $a, c > 0$, $ac > 0$ so
$$b^2 - 4ac < b^2 \quad |\sqrt{\ }$$
$$\sqrt{b^2 - 4ac} < \sqrt{b^2}$$
$$\sqrt{b^2 - 4ac} < -b \quad (\sqrt{b^2} = -b \text{ since } b < 0)$$
$$0 < -b - \sqrt{b^2 - 4ac} \quad |\div 2a, \text{ which is positive}$$

Then it follows that $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a} > 0$, too