Real Function Theory I — Handout
MAT 726, Spring 2016 — D. Ivanšić

Test Knowledge

Sections 2.7, 3.1–3.3, 4.1

Definitions

Cantor set (2.7)
Cantor-Lebesgue function (2.7)
Measurable function (3.1)
Characteristic function χ_A (3.2)
Simple function (3.2)
Riemann sum, Riemann integrability via Riemann sums (B&S 7.1)
Upper, lower Darboux sum (4.1)
Upper, lower Riemann integral (4.1)
Riemann integrable function via Darboux sums (4.1)

Theorems

Cantor set is closed, countable and has measure 0 (Prop. 2.19)
Cantor-Lebesgue function is increasing and continuous (Prop. 2.20)
Continuous bijection ψ maps a set of measure 0 to a set of nonzero measure, maps a measurable set to a nonmeasurable set (Prop. 2.21)
There exists a measurable set that is not Borel (Prop. 2.22)
Equivalence of measurable function definitions (Prop. 3.1)
Characteristic function χ_A is measurable iff A is measurable (3.2)
f is measurable iff $f^{-1}(U)$ is open for every open set U (Prop. 3.2)
f is measurable iff $f^{-1}(A)$ is open for every Borel set A (Prob. 3.7)
Continuous functions on measurable domains are measurable (Prop. 3.3)
Monotone functions on intervals are measurable (Prop. 3.4)
Function is measurable iff its restrictions are measurable (Prop. 3.5)
Lin. comb., products of measurable functions are measurable (Theorem 3.6)
Min, max, $|\ |$, $^+$, $^-$ of measurable functions are measurable (Prop. 3.8)
f continuous, g measurable $\implies$ $f \circ g$ is measurable (Prop. 3.7)
Composite of measurable functions need not be measurable (3.1)
Convergent sequence of measurable converges to measurable (Prop 3.9)
Simple Approximation Lemma (3.2)
f is measurable iff f is a limit of simple functions: Simple Approx. Thm. (3.2)
Littlewood's three principles (3.3)
A measurable set is nearly a finite union of open intervals (Theorem 2.12)
Pointwise convergence is nearly uniform: Egoroff's Theorem (3.3)
Every measurable function is nearly continuous: Lusin's Theorem (3.3)
Equiv. of Riemann integrability via Darboux or Riemann sums (Theorem 4.0)

Proofs

Cantor set is closed, countable and has measure zero (Prop. 2.19)
Simple Approximation Lemma (3.2)
Lemma 3.10
Every simple function is nearly continuous (Prop. 3.11)
Existence of a Riemann-nonintegrable function (4.1)
Examples where Riemann integral fails pointwise convergence (4.1)

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Sections 4.2–4.6

Definitions

Simple function and integral of a simple function (4.2)
Upper, lower integral of a measurable function over a set of finite measure (4.2)
Integral of a measurable function over a set of finite measure (4.2)
Integral of a nonnegative function (4.3)
Integrability of a nonnegative function (4.3)
f^+, f^-, integrability of a general function (4.4)
Uniform integrability of a family of functions (4.6)

Theorems

A Riemann integrable function is Lebesgue integrable (Theorem 4.3)
Linearity and Monotonicity of Integration (Prop. 4.2, Thms. 4.5, 4.10, 4.17)
Additivity of Integral (Coro. 4.6, Theorem 4.11, Coro. 4.18)
$|\int_E f| \leq \int_E |f|$ (Coro. 4.7, Prop. 4.16)
Uniform convergence theorem (Prop. 4.8)
Bounded Convergence Theorem (4.2)
Chebyshev's Inequality (4.3)
$f \geq 0$ and $\int_E f = 0 \implies f = 0$ ae on E (Prop. 4.9)
Fatou's Lemma (4.3)
Monotone Convergence Theorem (4.3)
Lebesgue Dominated Convergence Theorem (4.4)
General Lebesgue Dominated Convergence Theorem (Theorem 4.19)
Countable Additivity of Integration (Theorem 4.20)
Continuity of Integration (Theorem 4.21)
A finite collection of integrable functions is uniformly integrable (Prop. 4.23, 4.24)
Vitali Convergence Theorem (4.6)
Theorem 4.26

Proofs

Bounded Convergence Theorem (4.2)
Additivity of Integral (Coro. 4.6, Theorem 4.11, Coro. 4.18)
Chebyshev's Inequality (4.3)
$f \geq 0$ and $\int_E f = 0 \implies f = 0$ ae on E (Prop. 4.9)
Fatou's Lemma (4.3)
Examples where Fatou's Lemma has strict inequality
Examples of nonintegrable functions for which $\lim_{n\to\infty} \int_1^n f$ exists.

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Test Knowledge

Sections 7.1–7.3

Definitions

Essential upper bound, essentially bounded (7.1)
The spaces L^pE and l^p, $p \in [1, \infty]$ (7.1)
Norm on a linear space (7.1)
The space $C[a, b]$ and its norm $|| \, ||_{\max}$ (7.1)
The norm $|| \, ||_p$ on spaces L^pE and l^p (7.2)
The function f^* (Theorem 7.1)
Normed convergence of a sequence (7.3)
Cauchy sequence in a normed space (7.3)
Banach space (7.3)
Rapidly Cauchy sequence (7.3)

Theorems

$|a + b|^p \leq 2^p(|a|^p + |b|^p)$ (7.2)
Young's Inequality: $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$ (7.2)
Theorem 7.1, including Holder's inequality: $\int_E |f \cdot g| \leq ||f||_p \cdot ||g||_q$
Minkowski's inequality: $||f + g||_p \leq ||f||_p + ||g||_p$ (7.2)
$\mathcal{F}$ bounded in L^pE is uniformly integrable (Coro. 7.2)
$mE < \infty$ and $1 \leq p_1 < p_2 \leq \infty$ implies $L^{p_2}E \subset L^{p_1}E$, $||f||_{p_1} \leq c||f||_{p_2}$ (Coro. 7.3)
Convergent sequence is Cauchy (Prop 7.4)
Cauchy sequence is convergent if it has a convergent subsequence (Prop. 7.4)
Every rapidly Cauchy sequence is Cauchy (Prop. 7.5)
Every Cauchy sequence has a rapidly Cauchy subsequence (Prop. 7.5)
Every rapidly Cauchy sequence in L^pE converges wrt norm and pointwise (Thm 7.6)
Riesz-Fischer Theorem: L^pE is a Banach space (7.3)
Every norm-convergent sequence in L^pE
  has a subsequence that converges pointwise ae on E (7.3)
For $f_n, f \in L^pE$, if $f_n \to f$ pointwise ae on E, then
  $f_n \to f$ wrt norm iff $||f_n||_p \to ||f||_p$ (Theorem 7.7)

Proofs

Young's inequality (7.2)
Holder's inequality (Theorem 7.1)
$\mathcal{F}$ bounded in L^pE is uniformly integrable (Coro. 7.2)
Examples of functions in $L^{p_1}E$, but not in $L^{p_2}E$ (7.2)
Every rapidly Cauchy sequence is Cauchy (Prop. 7.5)
Every Cauchy sequence has a rapidly Cauchy subsequence (Prop. 7.5)
Examples of functions in L^pE, that converge pointwise, but not wrt norm (7.3)