Real Function Theory I — Exam 1 **MAT 726, Spring 2016 — D. Ivanšić**	**Name:** ______________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) State one of the four equivalent definitions of a measurable function.

Theory 2. (3pts) State the Simple Approximation Theorem.

Theory 3. (3pts) Define a lower Darboux sum.

Type A problems (5pts each)

A1. Let $f : E \to \mathbf{R}$ be a defined on a measurable set E so that $E = F \cup G$, where F and G are disjoint, $mG = 0$ and $f|_F : F \to \mathbf{R}$ is continuous. Show that $f : E \to \mathbf{R}$ is measurable.

A2. Given the function $f : (0, 1] \to \mathbf{R}$, $f(x) = \frac{1}{x}$, construct a sequence of step-functions that converges to f pointwise. A good picture with an explanation will suffice. (The existence of such a sequence is warranted by the Simple Approximation Theorem).

A3. Let $f : E \to \mathbf{R}$ be bounded and E measurable. Use the Simple Approximation Lemma to show there exists a sequence of functions $f_n : E \to \mathbf{R}$ such that $f_n \to f$ uniformly on E.

A4. Let $f : E \to \mathbf{R}$ be a simple function, $g : \mathbf{R} \to \mathbf{R}$ any function. Show that $g \circ f$ is a simple function. (Don't forget the part about measurability.)

A5. Let $f : [a, b] \to \mathbf{R}$ be bounded. Is there an upper bound for all upper Darboux sums $U(f, \mathcal{P})$, or a lower bound for all lower Darboux sums $L(f, \mathcal{P})$? Justify.

Type B problems (8pts each)

B1. Show that the Cantor set C has the property: for every $x, y \in C$, $x < y$, there exists a $t \notin C$ such that $x < t < y$. (Because of this, we say that C is *totally disconnected*.)

B2. Let $f : E \to \mathbf{R}$, where E is measurable, be a function such that $f^{-1}([a, b])$ is a measurable set for every $a, b \in \mathbf{R}$, $a < b$. Show that f is a measurable function.

B3. Let $f_n : E \to \mathbf{R}$ be a sequence of measurable functions. Show that the function $\sup f_n$ is measurable.

B4. Let $f_n : [0, 1] \to \mathbf{R}$ be defined by $f_n(x) = \begin{cases} nx, & \text{if } x \in [0, \frac{1}{n}] \\ 1, & \text{if } x \in (\frac{1}{n}, 1] \end{cases}$. Explain why $f_n \to f$ pointwise on $[0, 1]$, but not uniformly (what is f?). Given ϵ, determine the closed set F from Egoroff's theorem on which $f_n \to f$ uniformly on F, where $m([0, 1] - F) < \epsilon$. Good pictures with explanations will suffice.

B5. Let C be the Cantor set, and let $f : [0, 1] \to \mathbf{R}$ be defined by $f(x) = \begin{cases} x, & \text{if } x \notin C \\ 0, & \text{if } x \in C \end{cases}$. Show that f is a measurable function, and, given ϵ, determine the closed set F whose existence is guaranteed by Lusin's theorem, such that $m([0, 1]-F) < \epsilon$ and $f|_F$ is continuous.

B6. Prove that a bounded function $f : [a, b] \to \mathbf{R}$ is Riemann-integrable if and only if for every $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a, b]$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$.

Type C problems (12pts each)

C1. Let $\{q_n \mid n \in \mathbf{N}\}$ be an enumeration of rational numbers in $[0,1]$ and let a_n be a sequence whose limit is L and for which $|a_n| \leq M$, for all $n \in \mathbf{N}$. Show that the function $f : [0,1] \to \mathbf{R}$ defined by $f(x) = \begin{cases} a_n, & \text{if } x = q_n \\ L, & \text{if } x \notin \mathbf{Q} \cap [0,1] \end{cases}$ is Riemann-integrable by following the steps:

a) Given $\epsilon > 0$, show there exists an $n_0 \in \mathbf{N}$ such that $|a_n - L| < \epsilon$ and $\frac{4M}{n} < \epsilon$ for all $n \geq n_0$.

b) For an $n \geq n_0$, consider the partitition $\mathcal{P}$ of $[0,1]$ consisting of n^2 equal-width subintervals. Show that in at most $2n$ of those subintervals we have $M_i - m_i \leq 2M$, and that $M_i - m_i \leq 2\epsilon$ holds for the rest of the subintervals. Use this to show that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < 3\epsilon$.

c) Conclude that f is Riemann-integrable.

(Note: Thomae's function is a special case of this one.)

Real Function Theory I — Exam 2 MAT 726, Spring 2016 — D. Ivanšić

Name: ______________________________

Show all your work!

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) Let $f : E \to \overline{\mathbf{R}}$, $f \geq 0$. Define the integral of a nonnegative function.

Theory 2. (3pts) State Fatou's Lemma.

Theory 3. (3pts) State the Lebesgue Dominated Convergence Theorem.

Type A problems (5pts each)

A1. Determine if the function $f : [1, \infty) \to \mathbf{R}$, $f(x) = \frac{1}{x^3}$ is integrable over $[1, \infty)$. If it is, determine $\int_{[1,\infty)} f$. Justify your work with theory.

A2. Give an example of a sequence of functions $f_n : E \to \mathbf{R}$ such that $f_n \to f$ pointwise on E, but $\int_E f_n$ does not converge to $\int_E f$.

A3. Give a counterexample to the Bounded Convergence Theorem if we remove the assumption that $mE < \infty$, that is, give an example of a sequence of functions $f_n : E \to \mathbf{R}$, each with finite support, such that $|f_n| < M$ for all $n \in \mathbf{N}$, but $\int_E f_n$ does not converge to $\int_E f$.

A4. Suppose $f_n : E \to \overline{\mathbf{R}}$, $n \in \mathbf{N}$, $f : E \to \overline{\mathbf{R}}$ are all integrable over E and that $f_n \to f$ pointwise on E. Show that if $\int_E |f_n - f| \to 0$, then $\int_E f_n \to \int_E f$.

A5. Show that countable additivity of integration holds if $f \geq 0$, without the assumption that f is integrable.

A6. Let $\mathcal{F}$ and $\mathcal{G}$ be two uniformly integrable families of functions $f : E \to \overline{\mathbf{R}}$. Let $\alpha, \beta \in \mathbf{R}$ and define $\mathcal{H} = \{\alpha f + \beta g \mid f \in \mathcal{F}, g \in \mathcal{G}\}$. Show that the family of functions $\mathcal{H}$ is uniformly integrable.

Type B problems (8pts each)

B1. Let $f : [1, \infty) \to \mathbf{R}$, $f(x) = \frac{1}{n^2}$ for $x \in [n, n+\frac{1}{2})$, and $f(x) = -\frac{1}{3n^2}$, for $x \in [n+\frac{1}{2}, n+1)$, for every $n \in \mathbf{N}$. Determine whether f is integrable and, if it is, find its integral (expressed as a sum of a series). Justify your work with theory.

B2. Let $f : E \to \overline{\mathbf{R}}$, $f \geq 0$ be integrable. Given $\epsilon > 0$, show there exists a simple function ϕ of finite support, $0 \leq \phi \leq f$, such that $\int_E f - \int_E \phi < \epsilon$.

B3. Suppose $f_n : E \to \overline{\mathbf{R}}$, $f_n \geq 0$ on E, $n \in \mathbf{N}$, is a sequence of functions. Show the generalized Fatou Lemma: $\int_E \liminf f_n \leq \liminf \int_E f_n$.

B4. Suppose $f_n : E \to \overline{\mathbf{R}}$, $f_n \geq 0$ on E, $n \in \mathbf{N}$, is a *decreasing* sequence of functions that converges pointwise to $f : E \to \overline{\mathbf{R}}$. If f_1 is integrable, show that $\int_E f_n \to \int_E f$. Give an example where the conclusion does not hold if f_1 is not integrable.

B5. Give an example where countable additivity of integration fails, if f is not assumed to be integrable.

B6. Let $f : E \to \overline{\mathbf{R}}$ be integrable. Show that f^2 is integrable, where $f^2(x) = (f(x))^2$. (Hint: start with $\{x \in E \mid |f(x)| > 1\}$, apply Chebyshev's inequality, and then additivity over domains.)

Type C problems (12pts each)

C1. Let $\mathcal{F}$ be a uniformly integrable family of functions $f : E \to \overline{\mathbf{R}}$, and let $g : E \to \overline{\mathbf{R}}$ be a measurable function, and let $\mathcal{H} = \{fg \mid f \in \mathcal{F}\}$.
a) If g is bounded, show that $\mathcal{H}$ is uniformly integrable.
b) If g is integrable over E, does it follow that $\mathcal{H}$ is uniformly integrable?

C2. Let a bounded function $f : E \to \mathbf{R}$, $mE < \infty$, be integrable. The definition of integrability in this case does not assume that f is measurable. Show that f is measurable by using these steps:
a) Show there exist simple functions $\phi_n, \psi_n : E \to \mathbf{R}$ such that $\phi_n \leq f \leq \psi_n$ on E and $\int_E (\psi_n - \phi_n) < \frac{1}{2^{2n}}$, for every $n \in \mathbf{N}$.
b) Define $E_{mn} = \{x \in E \mid \psi_n(x) - \phi_n(x) > \frac{m}{2^n}\}$ and $F_m = \cup_{n \in \mathbf{N}} E_{mn}$. Use Chebyshev's inequality and countable additivity to show that $m(E_{mn}) < \frac{1}{2^n m}$ and $m(F_m) < \frac{1}{m}$.
c) Show that if $x \in F_m^c$, then $\psi_n(x) - \phi_n(x) \to 0$, so $\psi_n(x), \phi_n(x) \to f(x)$.
d) Observe that F_m is a descending sequence and show that $\psi_n - \phi_n \to 0$ ae on E.
e) Conclude that f is measurable function as the limit ae of simple, hence measurable, functions.

Real Function Theory I — Exam 3 **MAT 726, Spring 2016 — D. Ivanšić**

Name: ____________________

Show all your work!

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) If E is measurable, define the set L^pE.

Theory 2. (3pts) State Holder's inequality, along with the the statement about f^*.

Theory 3. (3pts) Define a Cauchy sequence in a normed linear space.

Type A problems (5pts each)

A1. On $C[a,b]$ define $||f||_1 = \int_{[a,b]} |f|$. Show that $||\ ||_1$ is a norm.

A2. For every $p \in [1,\infty]$, give an example of a function $f \in L^p[2,\infty)$ such that $f(x) > 0$ for all $x \in [2,\infty)$.

A3. Give an example of a function that is in $L^3(0,1)$, but is not in $L^5(0,1)$.

A4. If $f, g \in L^pE$, does it follow that the product fg is in L^pE?

A5. Let E be measurable, $mE < \infty$ and $S \subset L^1E$ be the subspace of simple functions with finite support. Is S a Banach space with respect to $||\ ||_1$?

A6. Let $f \in L^{p_1}E$, and let f be bounded. If $p_2 > p_1$, show that $f \in L^{p_2}E$. (This statement is different from the similar statement we had that assumed $mE < \infty$, but did not assume that f was bounded. Here, mE may be infinite. Don't do anything hard: this one needs only a little algebra.)

Type B problems (8pts each)

B1. Prove Holder's inequality for three functions: let $p, q, r \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$. If $f \in L^pE$, $g \in L^qE$ and $h \in L^rE$, then $\int_E |fgh| \leq ||f||_p||g||_q||h||_r$. (Start by using Holder's inequality on functions fg and h. First show that $fg \in L^{p'}E$, where $\frac{1}{p'} = \frac{1}{p} + \frac{1}{q}$.)

B2. Let $\mathcal{P}_n$ be the linear space of polynomials of degree $\leq n$, and $b_0, \dots, b_n$ a collection of $n+1$ distinct real numbers. Show that the functional $||\ || : \mathcal{P}_n \to \mathbf{R}$ is a norm on $\mathcal{P}_n$, where $||f|| = |f(b_0)| + |f(b_1)| + \cdots + |f(b_n)|$.

B3. Give an example of a convergent sequence $\{a_n \mid n \in \mathbf{N}\}$ of real numbers so that there does not exist a convergent series $\sum \epsilon_n$ satisfying $|a_k - a_{k+1}| \leq \epsilon_k$ for every $k \in \mathbf{N}$. (Your sequence cannot be monotone, since in this case convergence of $\{a_n\}$ is equivalent to convergence of the series $\sum |a_k - a_{k+1}|$, in which case you could use $\epsilon_k = |a_k - a_{k+1}|$.)

B4. Let $mE < \infty$ and $1 \leq p_1 < p_2 \leq \infty$. Corollary 7.3 states that for an $f \in L^{p_2}E$, there is a constant $c > 0$ such that $||f||_{p_1} \leq c||f||_{p_2}$. Show that there is no constant c satisfying $||f||_{p_2} \leq c||f||_{p_1}$ for all $f \in L^{p_2}E$, by examining the family of functions $\left\{ \frac{1}{x^{\frac{\alpha}{p_2}}} \mid 0 < \alpha < 1 \right\}$, on $E = (0,1]$.

B5. Let $f_n \to f$ pointwise on E, where $0 \le f_n \le f$ on E and $f \in L^p E$. Show that $f_n \in L^p E$ for every $n \in \mathbf{N}$, and that $f_n \to f$ in $L^p E$.

B6. Show that every Cauchy sequence has a rapidly Cauchy subsequence.

Type C problems (12pts each)

C1. Let $C \subset l^\infty$ be the subspace of all sequences that converge to a real number. Show that this is a Banach space with the norm $|| \, ||_\infty$.