Calculus 3 — Final Exam
MAT 309, Fall 2016 — D. Ivanšić

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Show all your work!

1. (12pts) Find the equation of the plane whose intersections with the x- and z-axes are 3 and 5, respectively, and that contains the point $(2, 3, -2)$.

Plane contains points:

$A = (3,0,0)$

$B = (0,0,5)$

$C = (2,3,-2)$

$AB = \langle -3, 0, 5 \rangle$

$AC = \langle -1, 3, -2 \rangle$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 0 & 5 \\ -1 & 3 & -2 \end{vmatrix} = -15\vec{i} - 11\vec{j} - 9\vec{k}$$

Take $\vec{n} = \langle 15, 11, 9 \rangle$

$15(x-3) + 11(y-0) + 9(z-0) = 0$

$15x + 11y + 9z = 45$ eq. of plane

2. (22pts) Let $f(x, y) = xy$.

a) Sketch the contour map for the function, drawing level curves for levels $k = -2, -1, 0, 1, 2$.

b) At point $(3, -4)$, find the directional derivative of f in the direction of $\langle 2, -1 \rangle$. In what direction is the directional derivative the greatest? What is the directional derivative in that direction?

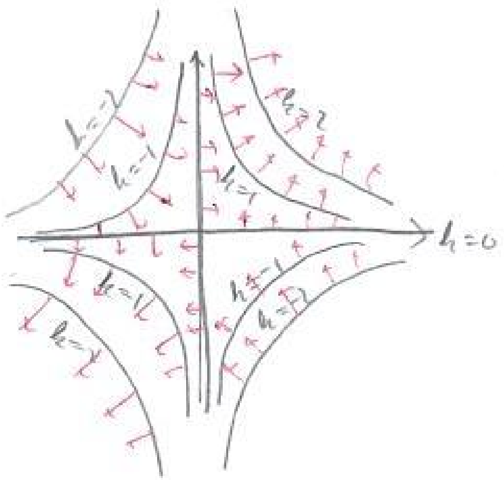
c) Let $\mathbf{F} = \nabla f$. Sketch the vector field $\mathbf{F}$.

d) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$ if C is the straight line from $(0, 0)$ to $(5, 2)$?

e) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$ if C is any part of one level curve?

a) $xy = k$

$y = \frac{k}{x}$



c) ∇f is perpendicular to level curves, in direction of increasing k

b) $\nabla f = \langle y, x \rangle$

$$D_u f(3,-4) = \nabla f \cdot \vec{u} = \langle -4, 3 \rangle \cdot \frac{1}{\sqrt{2^2+(-1)^2}} \langle 2, -1 \rangle$$

$$= \frac{1}{\sqrt{5}}(-8-3) = -\frac{11}{\sqrt{5}}$$

$D_u f$ greatest in direction of $\nabla f = \langle -4, 3 \rangle$

$D_u f$ in that direction is $\|\nabla f\| = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$

d) By fundamental theorem on line integrals

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A) = f(5,2) - f(0,0) = 10 - 0 = 10$$

e) $\int_C \nabla f \cdot d\vec{r} = \underbrace{f(B) - f(A)}_{\text{same, since both are on same level curve}} = 0$

3. (20pts) Find and classify the local extremes for $f(x, y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$.

$\nabla f = \langle 6xy - 12x,\ 3y^2 + 3x^2 - 12y \rangle$

$$\begin{cases} 6xy - 12x = 0 & \div 6 \\ 3y^2 + 3x^2 - 12y = 0 & \div 3 \end{cases}$$

$$\begin{cases} xy - 2x = 0 \Rightarrow x(y-2) = 0 \\ x^2 + y^2 - 4y = 0 \end{cases}$$

$\underline{x = 0}$ or $\underline{y = 2}$

$y^2 - 4y = 0$ $\qquad$ $x^2 + 4 - 8 = 0$

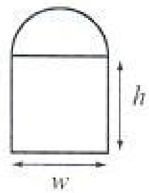
$y(y-4) = 0$ $\qquad$ $x^2 = 4$

$y = 0, 4$ $\qquad$ $x = \pm 2$

Candidates

Candidates	$D = \begin{vmatrix} 6y-12 & 6x \\ 6x & 6y-12 \end{vmatrix}$	
$(0,0)$	$\begin{vmatrix} -12 & 0 \\ 0 & -12 \end{vmatrix} = 144 > 0$	local max since $f_{xx} < 0$
$(0,4)$	$\begin{vmatrix} 12 & 0 \\ 0 & 12 \end{vmatrix} = 144 > 0$	local min since $f_{xx} > 0$
$(2,2)$	$\begin{vmatrix} 0 & 12 \\ 12 & 0 \end{vmatrix} = -144 < 0$	saddle pt
$(-2,2)$	$\begin{vmatrix} 0 & -12 \\ -12 & 0 \end{vmatrix} = -144 < 0$	saddle pt

4. (12pts) A Norman window is a rectangle topped by a semicircle. Use differentials to estimate how much the area of the window changes if the height increases from 52 to 56 inches, and the width decreases from 40 to 38 inches.



$$A = wh + \frac{1}{2}\pi \cdot \left(\frac{w}{2}\right)^2$$

$$= wh + \frac{\pi}{8}w^2$$

$$dA = \frac{\partial A}{\partial w}dw + \frac{\partial A}{\partial h}dh$$

$$= \left(h + \frac{\pi}{4}w\right)dw + w\,dh$$

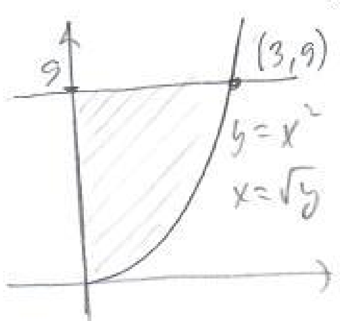
$$dw = -2,\quad dh = 4,\quad w = 40,\ h = 52$$

$$dA = \left(52 + \frac{\pi}{4}\cdot 40\right)\cdot(-2) + 40\cdot 4$$

$$= -104 - 20\pi + 160$$

$$= 56 - 20\pi \text{ in}^2$$

5. (18pts) Let D be the region bounded by the curves $y = x^2$, $x = 0$ and $y = 9$. Sketch the region and set up $\iint_D \frac{1}{y^{\frac{3}{2}}+1}\,dA$ as iterated integrals in both orders of integration. Then evaluate the double integral using the easier order. Sketch the region of integration first.



$$\text{Type 1}: \int_0^3\left(\int_{x^2}^9 \frac{1}{y^{\frac{3}{2}}+1}\,dy\right)dx$$

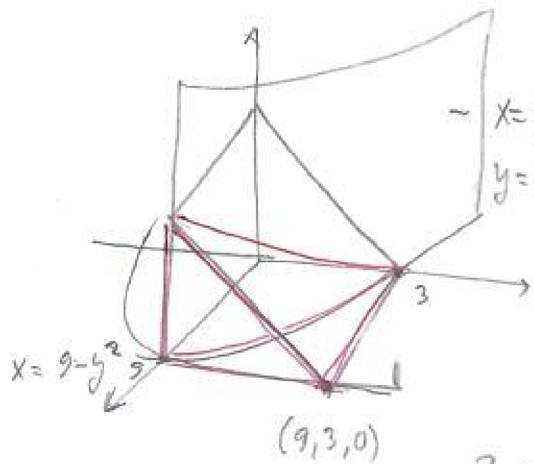
$$\text{Type 2} = \int_0^9\left(\int_0^{\sqrt{y}} \frac{1}{y^{\frac{3}{2}}+1}\,dx\right)dy \leftarrow \text{easier}$$

$$= \int_0^9 \frac{\sqrt{y}}{y^{\frac{3}{2}}+1}\,dy = \left[\begin{array}{ll} u = y^{\frac{3}{2}} & y=9,\ u=27 \\ du = \frac{3}{2}y^{\frac{1}{2}}dy & y=0,\ u=0 \\ \frac{2}{3}du = y^{\frac{1}{2}}dy & \end{array}\right]$$

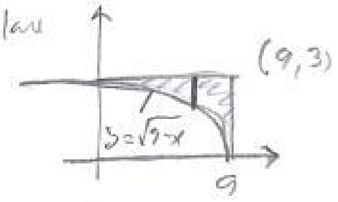
$$= \int_0^{27} \frac{1}{u+1}\,\frac{2}{3}du = \frac{2}{3}\ln(u+1)\Big|_0^{27} = \frac{2}{3}(\ln 28 - \ln 1)$$

$$= \frac{2\ln 28}{3}$$

6. (18pts) Sketch the region E that is bounded by the planes $z = 0$, $x = 9$, $z = 3 - y$ and the parabolic cylinder $x = 9 - y^2$. Then write the two iterated triple integrals that stand for $\iiint_E f\,dV$ which end in $dz\,dy\,dx$ and $dy\,dx\,dz$.



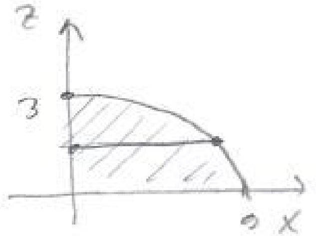
Projection to xy-plane



$$\int_0^9 \int_{\sqrt{9-x}}^{3} \int_0^{3-y} f \; dz\,dy\,dx$$

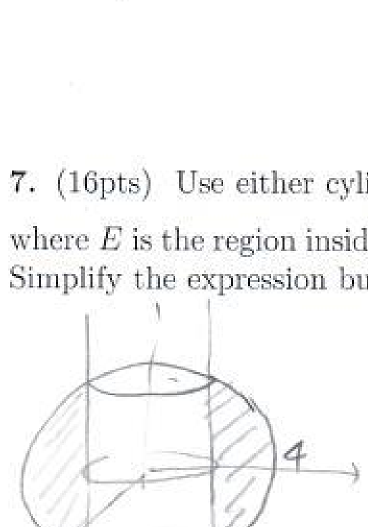
Projection to xz plane, need intersection of $z = 3 - y$ and $x = 9 - y^2$

$x = 9 - (3-z)^2 = 9 - (9 - 6z + z^2) = 6z - z^2$



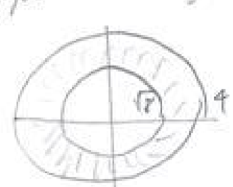
$$\int_0^3 \int_0^{6z-z^2} \int_{\sqrt{9-x}}^{3-z} f \; dy\,dx\,dx$$

7. (16pts) Use either cylindrical or spherical coordinates to set up $\displaystyle\iiint_E \frac{x+y}{x^2+y^2+z^2}\,dV$, where E is the region inside the sphere $x^2+y^2+z^2 = 16$ and outside the cylinder $x^2+y^2 = 8$. Simplify the expression but do NOT evaluate the integral. Sketch the region E.



Cylindrical coordinates: $x^2+y^2+z^2=16$, $r^2+z^2=16$, $z = \pm\sqrt{16-r^2}$

Projection to xy-plane



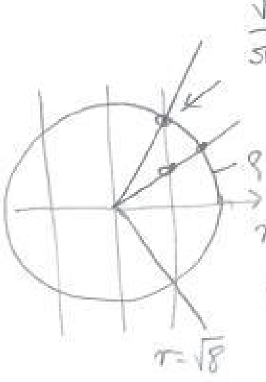
$$\int_0^{2\pi} \int_{\sqrt{8}}^{4} \int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} \frac{r^2(\cos\theta+\sin\theta)}{r^2+z^2}\, dz\,dr\,d\theta$$

$$\frac{x+y}{x^2+y^2+z^2}\cdot r = \frac{r\cos\theta + r\sin\theta}{r^2+z^2}\cdot r = \frac{r^2(\cos\theta+\sin\theta)}{r^2+z^2}$$

$\dfrac{\sqrt{8}}{\sin\phi} = 4$

$\sin\phi = \dfrac{\sqrt{8}}{4} = \dfrac{\sqrt{2}}{2}$

$\phi = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$



$\rho\sin\phi = \sqrt{8}$

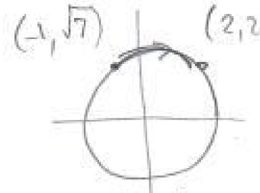
$\rho = \dfrac{\sqrt{8}}{\sin\phi}$

$$\frac{x+y}{x^2+y^2+z^2}\cdot\rho^2\sin\phi = \frac{\rho\sin\phi\cos\theta + \rho\sin\phi\cos\theta}{\rho^2}\cdot\rho^2\sin\phi$$

$$= \rho\sin^2\phi(\cos\theta+\sin\theta)$$

$$\int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{\frac{\sqrt{8}}{\sin\phi}}^{4} \rho\sin^2\phi\,(\cos\theta+\sin\theta)\, d\rho\,d\phi\,d\theta$$

8. (10pts) Set up and simplify the set-up, but do NOT evaluate the integral: $\int_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F}(x, y) = \langle xy, x + y^2 \rangle$, and C is the arc of the circle $x^2 + y^2 = 8$ from point $(-1, \sqrt{7})$ to point $(2, 2)$.



$x^2+y^2=8$

$y=\pm\sqrt{8-x^2}$

$x=t$

$y=\sqrt{8-t^2}$

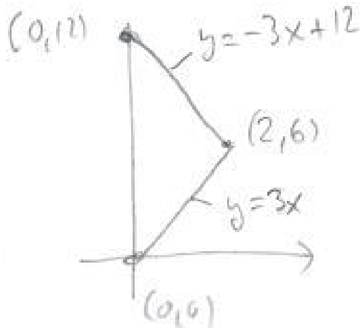
$$\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{2} \left\langle t\cdot\sqrt{8-t^2},\ t+8-t^2 \right\rangle \cdot \left\langle 1, \frac{-t}{\sqrt{8-t^2}} \right\rangle$$

$$= \int_{-1}^{2} t\sqrt{8-t^2} + \frac{-t^2}{\sqrt{8-t^2}} + (8-t^2)\cdot\frac{-t}{\sqrt{8-t^2}}\, dt$$

$$= \int_{-1}^{2} t\sqrt{8-t^2} - \frac{t^2}{\sqrt{8-t^2}} - t\sqrt{8-t^2}\, dt$$

$$= -\int_{-1}^{2} \frac{t^2}{\sqrt{8-t^2}}\, dt$$

9. (22pts) Use Green's theorem to find the line integral $\int_C (x^2+y^2)\,dx + xy\,dy$, where C is the boundary of the triangle with vertices $(0, 0)$, $(2, 6)$ and $(0, 12)$, traversed counterclockwise. Draw the triangle.



$\frac{12-6}{0-2} = \frac{6}{-2} = -3$

$y=-3x+12$

Green

$$\int_C x^2+y^2\,dx + xy\,dy = \iint_D \frac{\partial}{\partial x}xy - \frac{\partial}{\partial y}(x^2+y^2)\,dA$$

$$= \iint_D y - 2y\,dA = \int_0^2 \int_{3x}^{12-3x} -y\,dy\,dx$$

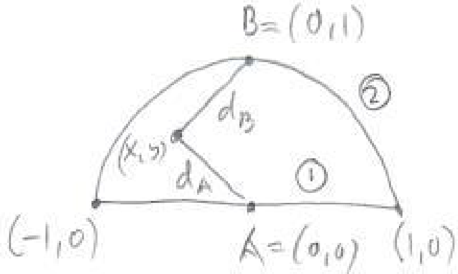
$$= \int_0^2 -\frac{y^2}{2}\Big|_{3x}^{12-3x} dx = -\frac{1}{2}\int_0^2 (12-3x)^2 - (3x)^2\,dx$$

$$= -\frac{1}{2}\int_0^2 144 - 72x + 9x^2 - 9x^2\,dx$$

$$= -\frac{72}{2}\int_0^2 2 - x\,dx = -36\left(2\cdot2 - \frac{x^2}{2}\Big|_0^2\right)$$

$$= -36\left(4 - \frac{1}{2}(4-0)\right) = -36\cdot 2 = -72$$

Bonus (10pts) Let $A = (0,0)$ and $B = (0,1)$, and let d_A and d_B represent the distance from a point (x,y) to A and B, respectively. Find the absolute maximum and minimum of $d_A^2 + d_B^2$ among all points (x,y) in the upper half of the unit disk $x^2+y^2 \le 1$, $y \ge 0$.



$$d_A^2 + d_B^2 = x^2+y^2+x^2+(y-1)^2$$

$$= 2x^2+2y^2-2y+1 = f(x,y)$$

$$\nabla f = \langle 4x, 4y-2 \rangle$$

$\nabla f = 0$ if $\begin{cases} 4x=0 \\ 4y-2=0 \end{cases}$ $\quad x=0$, $y=\frac{1}{2}$ $\quad (0,\frac{1}{2})$

Boundary: 1 $\quad x=t$, $y=0$, t in $[-1,1]$ $\quad g'(t)=0$, $4t=0$, $t=0$

$f(t,0) = 2t^2 = g(t)$

Candidates: $t=-1$, $t=0$, $t=1$

$(-1,0)$ $\quad (0,0)$ $\quad (1,0)$

2 $\quad x=\cos t$, $y=\sin t$, t in $[0,\pi]$

$f(\cos t, \sin t) = \underbrace{2\cos^2 t + 2\sin^2 t}_{=2} - 2\sin t + 1 = 3 - 2\sin t = g(t)$

$g'(t)=0$, $\quad -2\cos t = 0$, $\quad t=\frac{\pi}{2}$

$t=0$, $t=\frac{\pi}{2}$, $t=\pi$

$(1,0)$ $\quad (0,1)$ $\quad (-1,0)$

(x,y)	$f(x,y)$	
$(0,\frac{1}{2})$	$2\cdot\frac{1}{4} - 2\cdot\frac{1}{2}+1 = \frac{1}{2}$	min
$(-1,0)$	$2 \quad +1 = 3$	max
$(0,0)$	$1 = 1$	
$(1,0)$	$2 \quad +1 = 3$	max
$(0,1)$	$2-2 \quad +1 = 1$	