Calculus 3 — Exam 4
MAT 309, Fall 2016 — D. Ivanšić

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Show all your work!

1. (12pts) Let $\mathbf{F}(x,y) = \left\langle -\frac{x}{2}, -\frac{2y}{9} \right\rangle$.
a) Guess a function $f(x,y)$ so that $\mathbf{F} = \nabla f$.
b) Use the function f to draw the vector field without having to evaluate F at various points.

a) $f(x,y) = -\frac{x^2}{4} - \frac{y^2}{9}$

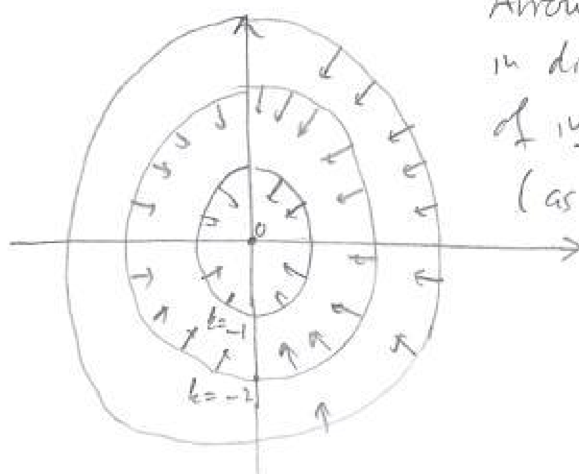
Level curves

$-\frac{x^2}{4} - \frac{y^2}{9} = c, \quad c \leq 0$

$\frac{x^2}{4} + \frac{y^2}{9} = -c$

$\frac{x^2}{-4c} + \frac{y^2}{-9c} = 1$

ellipses with semiaxes $2\sqrt{-c}, 3\sqrt{-c}$



Arrows point in direction of increase of c (as ∇f does)

2. (20pts) In both cases set up and simplify the set-up, but do NOT evaluate the integral.
a) $\int_C \frac{y}{x^2+y^2+z^2}\,ds$, where C is the helix $x = 3\cos t$, $y = \frac{t}{\pi}$, $z = 3\sin t$, t in $[0, 4\pi]$.

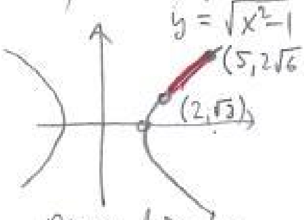
b) $\int_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F}(x,y) = \left\langle x^2y^2, \frac{x}{y} \right\rangle$, where C is the arc of the hyperbola $x^2 - y^2 = 1$ from point $(2, \sqrt{3})$ to point $(5, 2\sqrt{6})$.

a) $x = 3\cos t$
$y = \frac{t}{\pi}$
$z = 3\sin t$

$\int_C \frac{y}{x^2+y^2+z^2}\,ds = \int_0^{4\pi} \frac{\frac{t}{\pi}}{9\cos^2 t + \frac{t^2}{\pi^2} + 9\sin^2 t} \sqrt{(-3\sin t)^2 + \frac{1}{\pi^2} + (3\cos t)^2}$

$= \int_0^{4\pi} \frac{t}{\pi\left(9 + \frac{t^2}{\pi^2}\right)} \cdot \sqrt{9 + \frac{1}{\pi^2}}\,dt = \int_0^{4\pi} \frac{t}{\pi\left(9 + \frac{t^2}{\pi^2}\right)} \cdot \frac{\sqrt{9\pi^2+1}}{\pi}\,dt = \int_0^{4\pi} \frac{t\sqrt{9\pi^2+1}}{9\pi^2+t^2}\,dt$

b) $x^2 - y^2 = 1$
$y = \sqrt{x^2-1}$



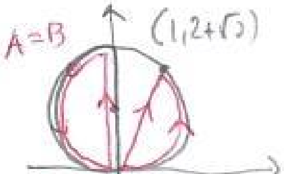
parametrization
$x = t$
$y = \sqrt{t^2-1}$ $\quad t$ in $[2,5]$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$

$\int_C \vec{F} \cdot d\vec{r} = \int_2^5 \left\langle t^2\left(\sqrt{t^2-1}\right)^2, \frac{t}{\sqrt{t^2-1}} \right\rangle \cdot \left\langle 1, \frac{t}{\sqrt{t^2-1}} \right\rangle dt$

$= \int_2^5 t^2(t^2-1) + \frac{t^2}{t^2-1}\,dt = \int_2^5 t^4 - t^2 + \frac{t^2}{t^2-1}\,dt$

3. (10pts) Let $f(x, y) = x^2 + xy + y^2$, and let $\mathbf{F} = \nabla f$. Apply the fundamental theorem for line integrals to answer:
a) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$ if C is the part of the right half of the circle $x^2 + (y-2)^2 = 4$ from $(0, 0)$ to $(1, 2+\sqrt{3})$? How about if C is a straight line segment from $(0, 0)$ to $(1, 2+\sqrt{3})$?
b) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$ if C is the curve consisting of the left half of the circle from a) together with the line segment from $(0, 0)$ to $(0, 4)$, traversed clockwise?



a) $\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$, hence

$$\int_C \nabla f \cdot d\vec{r} = f(1, 2+\sqrt{3}) - f(0,0) = 1 + 2 + \sqrt{3} + \underbrace{(2+\sqrt{3})^2}_{4+4\sqrt{3}+3} - 0 = 10 + 5\sqrt{3}$$

Integral is same for both curves, since their endpoints are same

b) $\int_C \nabla f \cdot d\vec{r} = f(B) - f(A) = 0$ since $A = B$

4. (22pts) a) Find curl of both of the vector fields below.
b) One of the fields is conservative. Find its potential function.

$\mathbf{F}(x, y, z) = \langle \sin z, x, x\cos z - \sin z \rangle$ $\qquad$ $\mathbf{G}(x, y, z) = \langle 3x^2y^2, 2x^3y + e^z, (2+y)e^z \rangle$

curl $\vec{F}$ =

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & x & x\cos z - \sin z \end{vmatrix}$$

$= (0-0)\vec{i} - (\cos z - \cos z)\vec{j} + (1-0)\vec{k}$

$= \vec{k} \neq \vec{0}$, so not conservative

curl $\vec{G}$ =

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y^2 & 2x^3y + e^z & (2+y)e^z \end{vmatrix}$$

$= (e^z - e^z)\vec{i} - (0-0)\vec{j} + (6x^2y - 6x^2y)\vec{k}$

$= \vec{0}$ since it is defined on all of $\mathbb{R}^3$, it is conservative.

Let $\nabla g = G$ $\quad \frac{\partial g}{\partial x} = 3x^2y^2$ so

$g = x^3y^2 + h(y,z)$

$2x^3y + e^z = \frac{\partial g}{\partial y} = \cancel{2x^3y} + \frac{\partial h}{\partial y}$

so $\frac{\partial h}{\partial y} = e^z$, $\quad h(y,z) = ye^z + k(z)$

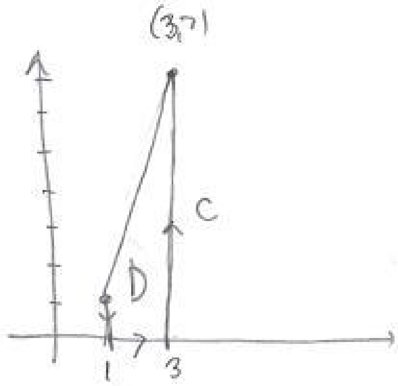
$g = x^3y^2 + ye^z + h(z)$

$(2+y)e^z = \frac{\partial g}{\partial z} = ye^z + h'(z)$

so $2e^z = h'(z)$, $\quad h(z) = 2e^z + C$

Thus $g(x,y,z) = x^3y^2 + ye^z + 2e^z + C$

5. (24pts) Use Green's theorem to find the line integral $\int_C (x^2+y^2)\,dx + xy\,dy$, where C is the boundary of the trapezoid with vertices $(1,0)$, $(3,0)$, $(3,7)$ and $(1,1)$, traversed counterclockwise. Draw the trapezoid.



$$\int_C (x^2+y^2)\,dx + xy\,dy = \left[\begin{array}{c}\text{Green's}\\ \text{Theorem}\end{array}\right]$$

$$= \iint_D \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2+y^2)\,dA$$

$$= \iint_D y - 2y\,dA = -\iint_D y\,dA$$

slope $= \frac{7-1}{3-1} = 3$

$y-1 = 3(x-1)$

$y = 3x-2$

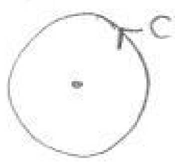
$$= -\int_1^3 \int_0^{3x-2} y\,dy\,dx = -\int_1^3 \frac{1}{2}y^2\Big|_0^{3x-2} dx$$

$$= -\frac{1}{2}\int_1^3 (3x-2)^2 - 0\,dx = -\frac{1}{2}\,\frac{(3x-2)^3}{3\cdot 3}\Big|_1^3$$

$$= -\frac{1}{18}\left(7^3 - 1^3\right) = -\frac{342}{18} = -\frac{38}{2} = -19$$

$343-1$

6. (12pts) Use Green's theorem to find the area of the unit disk.



C: $x = \cos t$, $y = \sin t$, t in $[0, 2\pi]$

$$\iint 1\,dA = \int_C x\,dy = \int_0^{2\pi} \cos t \cdot \cos t\,dt$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \int_0^{2\pi} \frac{1+\cos(2t)}{2}\,dt \qquad \int_0^{2\pi} = 0$$

$$= \frac{1}{2}\cdot 2\pi = \pi$$

or $$\iint 1\,dA = \frac{1}{2}\int_C -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} -\sin t(-\sin t) + \cos t\cdot\cos t\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = \frac{1}{2}\int_0^{2\pi} 1\,dt = \frac{1}{2}\cdot 2\pi = \pi$$

Bonus. (10pts) Recall that we have shown that the field $\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$, defined on the region $D = \mathbf{R}^2$ without the origin, is not conservative on D, since its line integral over a unit circle is not 0.

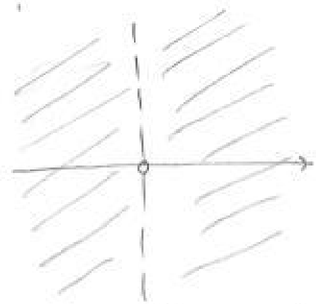
a) Let $f(x, y) = \arctan \frac{y}{x}$. Show that $\nabla f = \mathbf{F}$. Recall that $(\arctan u)' = \frac{1}{1+u^2}$.

b) Why does a) not contradict our earlier finding of $\mathbf{F}$ not being conservative on D?

a) $$\nabla \arctan \frac{y}{x} = \left\langle \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right), \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} \right\rangle$$

$$= \left\langle -\frac{y}{x^2+y^2}, \frac{1}{x+\frac{y^2}{x}} \cdot \frac{x}{x} \right\rangle = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

b) The domain of $\arctan \frac{y}{x}$ is : Can't have $x=0$



This is not all of D, hence this f is not a potential function for $\vec{F}$ on its entire domain. Therefore, no contradiction.