Calculus 3 — Exam 3
MAT 309, Fall 2016 — D. Ivanšić

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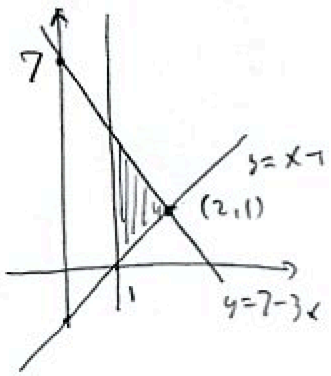
1. (18pts) Find $\iint_D \frac{1}{x}\, dA$ if D is the triangle bounded by lines $x = 1$, $y = x - 1$ and $y = 7 - 3x$. Sketch the region of integration first.



$7-3x = x-1$

$8 = 4x$

$x = 2$

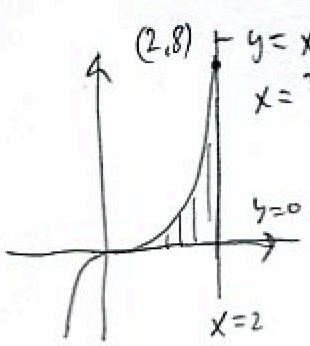
$$\iint_D \frac{1}{x}\,dA = \int_1^2 \left(\int_{x-1}^{7-3x} \underbrace{\frac{1}{x}}_{\text{constant}}\, dy \right) dx = \int_1^2 \frac{1}{x}(7-3x-(x-1))\,dx$$

$$= \int_1^2 \frac{1}{x}(8-4x)\,dx = \int_1^2 \frac{8}{x} - 4\,dx$$

$$= 8\ln x \Big|_1^2 - 4(2-1)$$

$$= 8(\ln 2 - \underbrace{\ln 1}_{=0}) - 4 = 8\ln 2 - 4$$

2. (18pts) Let D be the region bounded by the curves $y = x^3$, $x = 2$ and $y = 0$. Sketch the region and set up $\displaystyle\iint_D \frac{1}{(x^4+1)^2}\, dA$ as iterated integrals in both orders of integration. Then evaluate the double integral using the easier order. Sketch the region of integration first.



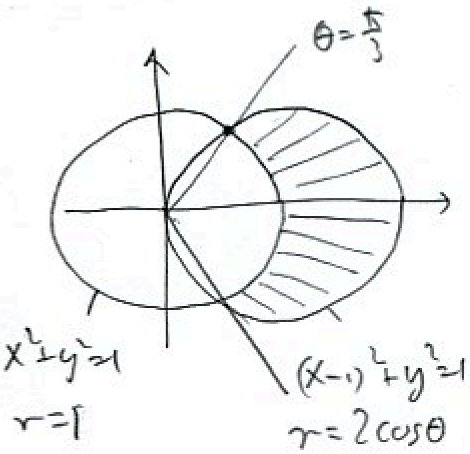
$$\int_0^2 \left(\int_0^{x^3} \frac{1}{(x^4+1)^2}\, dy \right) dx \quad \text{or} \quad \int_0^8 \left(\int_{\sqrt[3]{y}}^2 \frac{1}{(x^4+1)^2}\, dx \right) dy$$

easier

$$= \int_0^2 \frac{1}{(x^4+1)^2}\cdot(x^3-0)\,dx = \int_0^2 \frac{x^3}{(x^4+1)^2}\,dx = \left[\begin{array}{ll} u = x^4+1 & x=2,\ u=17 \\ du = 4x^3\,dx & x=0 \\ \frac{1}{4}du = x^3\,dx & u=1 \end{array}\right]$$

$$= \int_1^{17} \frac{1}{u^2}\cdot\frac{1}{4}\,du = \frac{1}{4}\left(-\frac{1}{u}\right)\Big|_1^{17} = \frac{-1}{4}\left(\frac{1}{17}-1\right) = \frac{1}{4}\cdot\frac{16}{17} = \frac{4}{17}$$

3. (18pts) Use polar coordinates to find $\iint_D \frac{x}{x^2+y^2}\,dA$, if D is the region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$. Sketch the region of integration first.



bending factor

$$\iint_D \frac{x}{x^2+y^2}\,dA = \int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} \frac{r\cos\theta}{r^2}\cdot r\,dr\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} \cos\theta\,dr\,d\theta = \int_{-\pi/3}^{\pi/3} \cos\theta(2\cos\theta-1)\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} 2\cos^2\theta - \cos\theta\,d\theta = \int_{-\pi/3}^{\pi/3} 1+\cos(2\theta)-\cos\theta\,d\theta$$

intersection:

$1 = 2\cos\theta$

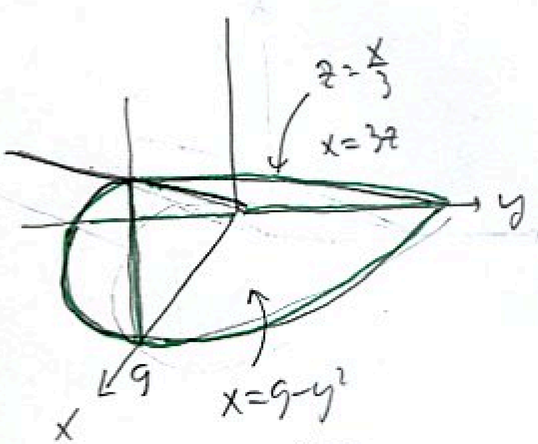
$\cos\theta = \frac{1}{2}$

$\theta = \pm\frac{\pi}{3}$

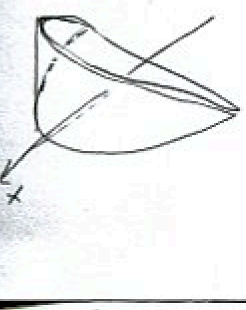
$$= 1\cdot\left(\frac{\pi}{3}-\left(-\frac{\pi}{3}\right)\right) + \left.\frac{\sin(2\theta)}{2}\right|_{-\pi/3}^{\pi/3} - \left.\sin\theta\right|_{-\pi/3}^{\pi/3} = \frac{2\pi}{3} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}-\left(-\frac{\sqrt{3}}{2}\right)\right) - \left(\frac{\sqrt{3}}{2}-\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= \frac{2\pi}{3} - \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}\sqrt{3}}{2} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

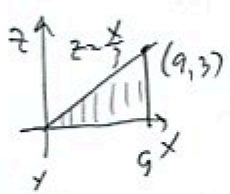
4. (18pts) Sketch the region E that is bounded by the planes $z=0$, $z=\frac{x}{3}$ and the parabolic cylinder $x=9-y^2$. Then write the two iterated triple integrals that stand for $\iiint_E f\,dV$ which end in $dy\,dz\,dx$ and $dx\,dz\,dy$.



$y = \pm\sqrt{9-x}$



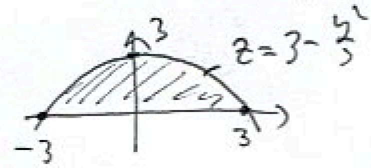
Projection to xz plane:



$$\int_0^9\int_0^{\frac{x}{3}}\int_{-\sqrt{9-x}}^{\sqrt{9-x}} f\,dy\,dz\,dx$$

Projection to yz plane:

$$\begin{cases} z = \frac{x}{3} \\ x = 9-y^2 \end{cases} \qquad 3z = 9-y^2 \qquad z = 3-\frac{y^2}{3}$$



$$\int_{-3}^{3}\int_{0}^{3-\frac{y^2}{3}}\int_{3z}^{9-y^2} f\,dx\,dz\,dy$$

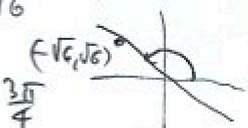
5. (10pts) Find the cylindrical and spherical coordinates of the point whose cartesian coordinates are $(-\sqrt{6}, \sqrt{6}, 2)$.

$$r = \sqrt{(-\sqrt{6})^2 + \sqrt{6}^2} = \sqrt{12} = 2\sqrt{3}$$

$$\rho = \sqrt{(-\sqrt{6})^2 + \sqrt{6}^2 + 2^2} = \sqrt{16} = 4$$

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{6}}{-\sqrt{6}} = -1$$

$$\theta = -\frac{\pi}{4} + \underbrace{\pi}_{\text{due to quadrant}} = \frac{3\pi}{4}$$



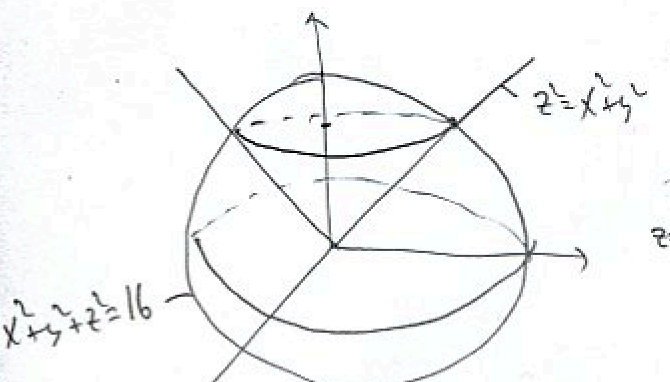
$$\cos\phi = \frac{z}{\rho} = \frac{2}{4} = \frac{1}{2}$$

$$\phi = \frac{\pi}{3}$$

Cylindrical: $(r, \theta, z) = (2\sqrt{3}, \frac{3\pi}{4}, 2)$

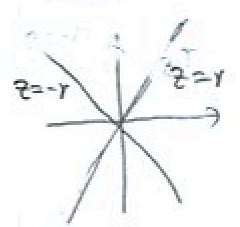
Spherical: $(\rho, \theta, \phi) = (4, \frac{3\pi}{4}, \frac{\pi}{3})$

6. (18pts) Use either cylindrical or spherical coordinates to set up $\iiint_E x^2 + z^2\, dV$, where E is the region inside the sphere $x^2 + y^2 + z^2 = 16$ and above the cone $x^2 + y^2 = z^2$. Simplify the expression but do not evaluate the integral. Sketch the region E.



$z^2 = x^2 + y^2$

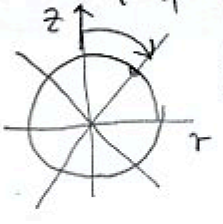
$z = \pm r$



Spherical:

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \left((\rho\sin\phi\cos\theta)^2 + (\rho\cos\phi)^2\right)\rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \rho^4\left(\sin^3\phi\cos^2\theta + \cos^2\phi\sin\phi\right) d\rho\, d\phi\, d\theta$$



$$\begin{cases} z^2 = r^2 \\ r^2 + z^2 = 16 \end{cases}$$

$2r^2 = 16$

$r^2 = 8$

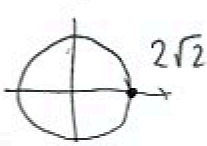
$r = 2\sqrt{2}$

Cylindrical:

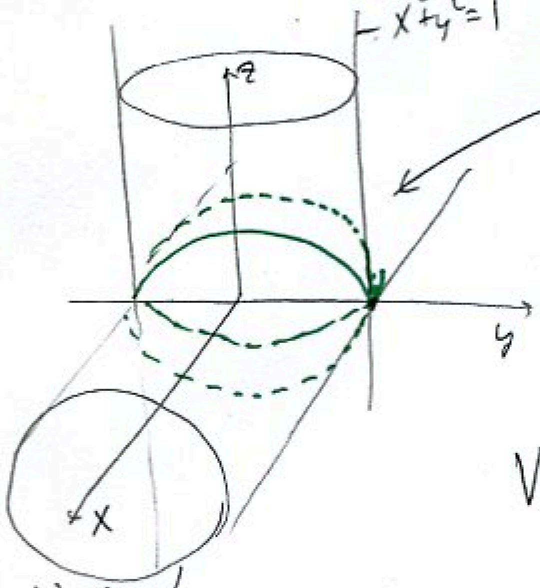
$$\int_0^{2\pi}\int_0^{2\sqrt{2}}\int_r^{\sqrt{16-r^2}} \left((r\cos\theta)^2 + z^2\right) r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2\sqrt{2}}\int_r^{\sqrt{16-r^2}} r^3\cos^2\theta + rz^2 \, dz\, dr\, d\theta$$

Projection to xy plane:

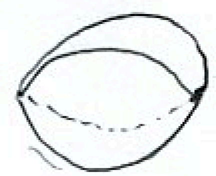


Bonus (10pts) Find the volume of the solid that is inside both cylinders: $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$.

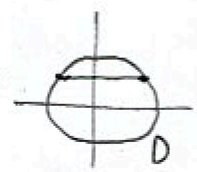


$y^2+z^2=1$

$z = \pm\sqrt{1-y^2}$



Projection to xy-plane is a disk:



$$V = \iiint_E 1 \, dV = \iint_D \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1 \, dz \, dA$$

$$= \iint_D 2\sqrt{1-y^2} \, dA \leftarrow \text{better to integrate by } x \text{ first}$$

$$= \int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 2\sqrt{1-y^2} \, dx \right) dy$$

$$= \int_{-1}^{1} 2\sqrt{1-y^2} \left(\sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right) \right) dy$$

$$= \int_{-1}^{1} 4(1-y^2) \, dy = 4\cdot(1-(-1)) - \frac{y^3}{3}\Big|_{-1}^{1}$$

$$= 8 - \frac{1}{3}(1-(-1)) = 8 - \frac{2}{3} = \frac{22}{3}$$