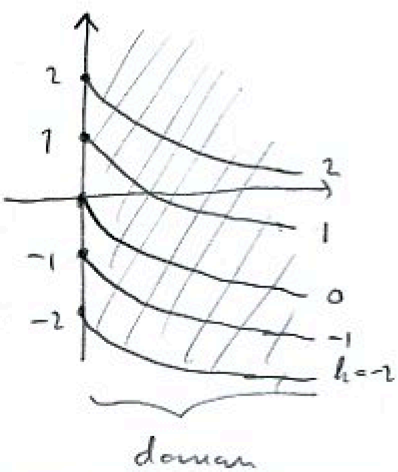
Calculus 3 — Exam 2 **MAT 309, Fall 2016 — D. Ivanšić**	**Name:** Saul Ocean *Show all your work!*

1. (22pts) Let $h(x,y) = \sqrt{x} + y$.
a) Find the domain of h.
b) Sketch the contour map for the function, drawing level curves for levels $k = -2, -1, 0, 1, 2$. Note the domain on the picture.
c) At point $(9,-2)$, find the directional derivative of f in the direction of $\langle 1, -2\rangle$. In what direction is the directional derivative the greatest? What is the directional derivative in that direction?
d) Suppose the surface $z = \sqrt{x}+y$ represents a mountain (x, y, z in kilometers), and assume a climber is passing through point $(9,-2,1)$ so that the horizontal component of her motion is in direction of $\langle 1,-2\rangle$ with unit speed (time in hours). Is she ascending or descending, and at which rate? What are the units?

a) Must have
$x \geq 0$

b) $\sqrt{x}+y = k$
$y = -\sqrt{x} + k$



c) $\nabla h = \langle \frac{1}{2\sqrt{x}}, 1\rangle$ $\quad \nabla h(9,-2) = \langle \frac{1}{6}, 1\rangle$

$\vec{u} = \frac{1}{\sqrt{1^2+(-2)^2}}\langle 1,-2\rangle = \frac{1}{\sqrt{5}}\langle 1,-2\rangle$

$D_{\vec{u}}h = \langle \frac{1}{6}, 1\rangle \cdot \frac{1}{\sqrt{5}}\langle 1,-2\rangle = \frac{1}{\sqrt{5}}\left(\frac{1}{6}-2\right) = -\frac{11}{6\sqrt{5}}$

$D_{\vec{u}}h$ is greatest when $\vec{u}$ is in direction of $\nabla h = \langle \frac{1}{6}, 1\rangle$, and equals $|\langle \frac{1}{6}, 1\rangle| = \sqrt{\frac{1}{36}+1} = \frac{\sqrt{37}}{6}$

d) Since $D_{\vec{u}}h = -\frac{11}{6\sqrt{5}} < 0$, she is descending at $\frac{11}{6\sqrt{5}}$ km/hr.

2. (12pts) Find the equation of the tangent plane to the hyperboloid of one sheet $x^2 + 2y^2 - z^2 = 23$ at the point $(4,-2,-1)$. Simplify the equation to standard form.

$F(x,y,z) = x^2+2y^2-z^2-23$

$\nabla F = \langle 2x, 4y, 2z\rangle$

$\nabla F(4,-2,-1) = \langle 8,-8,2\rangle$

Use $\langle 4,-4,1\rangle$ as normal vector

Equation of plane:

$4(x-4) - 4(x-(-2)) + (z-(-1)) = 0$

$4x - 4y + z - 16 - 8 + 1 = 0$

$4x - 4y + z = 23$

3. (18pts) Let $K = ye^{-x^2-y^2}$, $x = \dfrac{\sin u}{\cos v}$, $y = \cos u + \sin v$. Use the chain rule to find $\dfrac{\partial K}{\partial v}$ when $u = \frac{\pi}{2}$, $v = \frac{\pi}{4}$.

$$\frac{\partial K}{\partial v} = \frac{\partial K}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial K}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= y e^{-x^2-y^2}(-2x)\cdot \sin u \cdot\left(\frac{\sin v}{\cos^2 v}\right) + \left(e^{-x^2-y^2} + y e^{-x^2-y^2}(-2y)\right)\cos v$$

$$= e^{-x^2-y^2}\left(-\frac{2xy\sin u \sin v}{\cos^2 v} + (1-2y^2)\cos v\right)$$

$$= e^{-2-\frac{1}{2}}\left(-\frac{2\cdot\sqrt{2}\cdot\frac{\sqrt{2}}{2}\cdot 1\cdot\frac{\sqrt{2}}{2}}{\frac{1}{2}} + \underbrace{\left(1-2\left(\frac{\sqrt{2}}{2}\right)^2\right)}_{=0}\frac{\sqrt{2}}{2}\right)$$

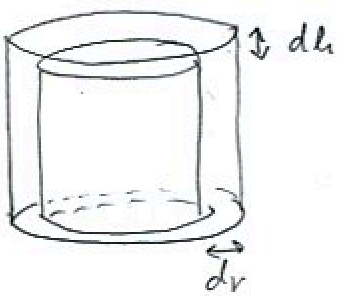
$$= e^{-\frac{5}{2}}\cdot(-2\sqrt{2}) = -2\sqrt{2}\,e^{-\frac{5}{2}}$$

when $u = \frac{\pi}{2}, v = \frac{\pi}{4}$

$x = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}$

$y = 0 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$

4. (14pts) A museum is housed in a cylindrical building (with a flat roof) with interior diameter 40 meters and height 30 meters. Use differentials to estimate the amount of concrete used to build this museum, if its walls have thickness 0.3 meters and the roof has thickness 0.1 meter.



$r = 20$

$h = 30$

$$V = \pi r^2 h$$

$$\Delta V \approx dV = V_r\,dr + V_h\,dh$$

$$dV = \pi\cdot 2r\cdot h\,dr + \pi r^2\,dh$$

$$= \pi\left(2\cdot 20\cdot 30\cdot 0.3 + 20^2\cdot 0.1\right)$$

$$= \pi\left(1200\cdot 0.3 + 400\cdot 0.1\right)$$

$$= \pi(360 + 40) = 400\pi \text{ m}^3$$

5. (14pts) Use implicit differentiation to find $\frac{\partial z}{\partial y}$ at the point $(3, -2, 4)$, if $x\ln(x^2+yz) + y\sqrt{7z-4x} = -8$.

$$F(x,y,z) = x\ln(x^2+yz) + y\sqrt{7z-4x} - 8$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{x\cdot\frac{1}{x^2+yz}\cdot z + \sqrt{7z-4x}}{x\cdot\frac{1}{x^2+yz}\cdot y + y\frac{1}{2\sqrt{7z-4x}}\cdot 7}$$ eval. at $(3,-2,4)$

$$\frac{\partial z}{\partial y} = -\frac{3\cdot\frac{1}{9-8}\cdot 4 + \sqrt{28-12}}{3\cdot\frac{1}{9-8}\cdot(-2) + (-2)\frac{1}{2\sqrt{28-12}}\cdot 7} = -\frac{12+4}{-6-\frac{14}{8}} = -\frac{16}{-\frac{31}{4}} = \frac{64}{31}$$

$\frac{7}{4}$

6. (20pts) Find and classify the local extremes for $f(x,y) = x^3 - 12xy + 8y^3$.

$$\nabla f = \langle 3x^2 - 12y, -12x + 24y^2\rangle$$

$$D(x,y) = \begin{vmatrix} 6x & -12 \\ -12 & 48y \end{vmatrix}$$

$$\begin{cases} 3x^2 - 12y = 0 \\ -12x + 24y^2 = 0 \end{cases}$$

$$D(0,0) = \begin{vmatrix} 0 & -12 \\ -12 & 0 \end{vmatrix} = -144 < 0$$ so saddle point

$$\begin{cases} x^2 - 4y = 0 \Rightarrow y = \frac{x^2}{4} \\ -x + 2y^2 = 0 \end{cases}$$ put in 2nd eq.

$$D(2,1) = \begin{vmatrix} \underset{>0}{12} & -12 \\ -12 & 48 \end{vmatrix} = 12\cdot 48 - 12^2 > 0$$ is a local min.

$$-x + 2\cdot\left(\frac{x^2}{4}\right)^2 = 0$$

$$-x + \frac{2x^4}{16} = 0 \quad | \cdot$$

$$-x + \frac{x^4}{8} = 0 \quad | \cdot 8$$

$f(0,0) = 0$

$f(2,1) = 8 - 24 + 8 = -8$

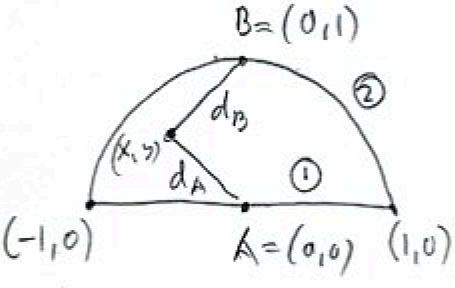
$$x(x^3 - 8) = 0$$

$x = 0$ or 2

Crit. pts: $(0,0), (2,1)$

so $y = 0$ or 1

Bonus (10pts) Let $A = (0,0)$ and $B = (0,1)$, and let d_A and d_B represent the distance from a point (x, y) to A and B, respectively. Find the absolute maximum and minimum of $d_A^2 + d_B^2$ among all points (x, y) in the upper half of the unit disk $x^2 + y^2 \leq 1$, $y \geq 0$.



$$d_A^2 + d_B^2 = x^2 + y^2 + x^2 + (y-1)^2$$

$$= 2x^2 + 2y^2 - 2y + 1 = f(x,y)$$

$$\nabla f = \langle 4x, 4y-2 \rangle$$

$\nabla f = 0$ if $\begin{cases} 4x = 0 \\ 4y - 2 = 0 \end{cases}$ $\quad x = 0$, $y = \frac{1}{2}$ $\quad (0, \frac{1}{2})$

Boundary: ① $x = t$, $y = 0$, t in $[-1, 1]$

$f(t, 0) = 2t^2 = g(t)$

$g'(t) = 0$, $4t = 0$, $t = 0$

Candidates: $t = -1$, $t = 0$, $t = 1$

$(-1,0)$ $(0,0)$ $(1,0)$

② $x = \cos t$, $y = \sin t$, t in $[0, \pi]$

$f(\cos t, \sin t) = \underbrace{2\cos^2 t + 2\sin^2 t}_{=2} - 2\sin t + 1 = 3 - 2\sin t = g(t)$

$g'(t) = 0$, $-2\cos t = 0$, $t = \frac{\pi}{2}$

$t = 0$, $t = \frac{\pi}{2}$, $t = \pi$

$(1,0)$ $(0,1)$ $(-1,0)$

(x,y)	$f(x,y)$	
$(0, \frac{1}{2})$	$2 \cdot \frac{1}{4} - 2 \cdot \frac{1}{2} + 1 = \frac{1}{2}$	min
$(-1,0)$	$2 \quad +1 = 3$	max
$(0,0)$	$1 = 1$	
$(1,0)$	$2 \quad +1 = 3$	max
$(0,1)$	$2 - 2 + 1 = 1$	