Calculus 3 — Exam 1
MAT 309, Fall 2016 — D. Ivanšić

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1. (13pts) Let $\mathbf{u} = \langle 2, 3, -5\rangle$ and $\mathbf{v} = \langle 0, -3, 1\rangle$.
a) Calculate $2\mathbf{u}$, $\mathbf{u} - 2\mathbf{v}$, and $\mathbf{u}\cdot\mathbf{v}$.
b) Find the unit vector in direction of $\mathbf{v}$.
c) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

a) $2\vec{u} = \langle 4, 6, -10\rangle$

$\vec{u} - 2\vec{v} = \langle 2, 3, -5\rangle - 2\langle 0, -3, 1\rangle$

$= \langle 2, 3, -5\rangle - \langle 0, -6, 2\rangle$

$= \langle 2, 9, -7\rangle$

$\vec{u}\cdot\vec{v} = 2\cdot 0 + 3\cdot(-3) + (-5)\cdot 1 = -14$

b) $|\vec{v}| = \sqrt{0^2 + (-3)^2 + 1^2} = \sqrt{10}$

$\vec{u} = \frac{1}{\sqrt{10}}\langle 0, -3, 1\rangle$

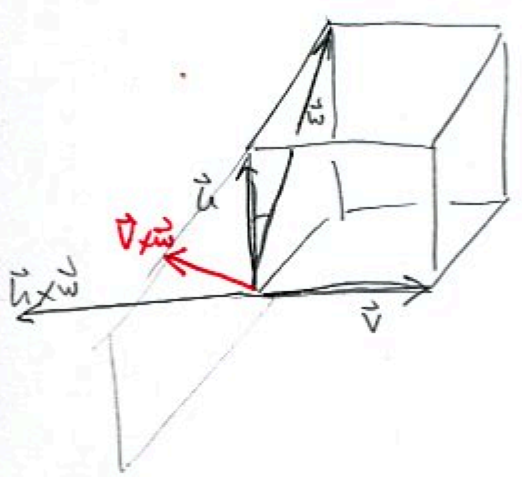
c) $\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$

$= \frac{-14}{\sqrt{10}^2}\langle 0, -3, 1\rangle$

$= -\frac{7}{5}\langle 0, -3, 1\rangle$

$= \langle 0, \frac{21}{5}, -\frac{7}{5}\rangle$

2. (13pts) In the picture, vectors $\mathbf{u}$ and $\mathbf{v}$ are on sides of a cube with side-length 1, and $\mathbf{w}$ is the diagonal of one of the sides. Draw the vectors $\mathbf{u}\times\mathbf{w}$ and $\mathbf{v}\times\mathbf{w}$ and determine their length.



$|\vec{u}\times\vec{w}| = |\vec{u}||\vec{w}|\sin\frac{\pi}{4} = 1\cdot\sqrt{2}\cdot\frac{\sqrt{2}}{2} = 1$

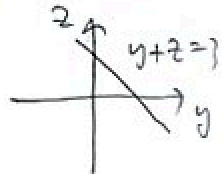
$\vec{u}\times\vec{w}$ is in direction opposite of $\vec{v}$

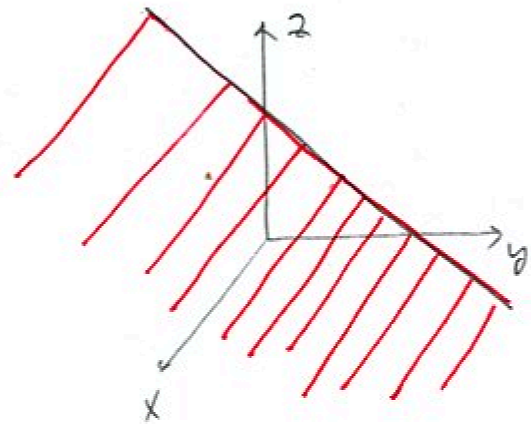
$|\vec{v}\times\vec{w}| = |\vec{v}||\vec{w}|\sin\frac{\pi}{2} = 1\cdot\sqrt{2}\cdot 1 = \sqrt{2}$

$\vec{v}\times\vec{w}$ is in the plane spanned by $\vec{u}, \vec{w}$

3. (8pts) Draw the region in $\mathbf{R}^3$ described by:

$y + z = 3,\ x \geq 0$





Half-plane
(due to $x \geq 0$)
based on line $y+z=3$

4. (12pts) Find the equation of the plane that contains the line $x = 3t$, $y = 7 - 2t$, $z = -4 + 2t$ and the point $(-2, 3, -4)$.

$x = 3t$

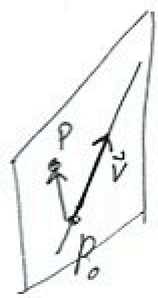
$y = 7 - 2t$

$z = -4 + 2t$

$P_0 = \langle 0, 7, -4)$

$P = (-2, 3, -4)$

$\vec{P_0P} = \langle -2, -4, 0 \rangle$



normal vector is

$$-\vec{P_0P} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 4 & 0 \\ 3 & -2 & 2 \end{vmatrix} = 8\vec{i} - 4\vec{j} + (-4-12)\vec{k}$$

$$= 8\vec{i} - 4\vec{j} - 16\vec{k},$$

take $\vec{n} = 2\vec{i} - \vec{j} - 4\vec{k}$

$$2(x-(-2)) - (y-3) - 4(z-(-4)) = 0$$

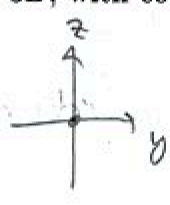
$$2x - y - 8z - 9 = 0$$

5. (16pts) This problem is about the surface $x + y^2 + 3z^2 = 0$.
a) Identify and sketch the intersections of this surface with the coordinate planes.
b) Sketch the surface in 3D, with coordinate system visible.

a) $x = 0$

$y^2 + 3z^2 = 0$

point $(0,0)$

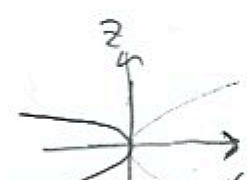


$y = 0$

$x + 3z^2 = 0$

$x = -3z^2$

a parabola

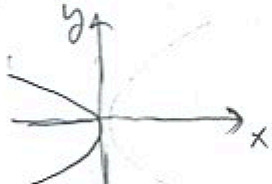


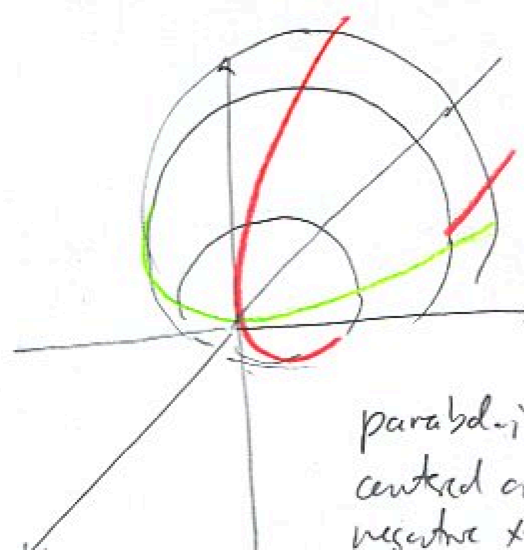
$z = 0$

$x + y^2 = 0$

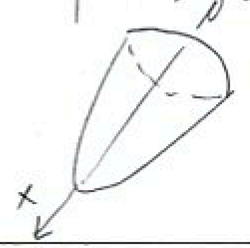
$x = -y^2$

a parabola





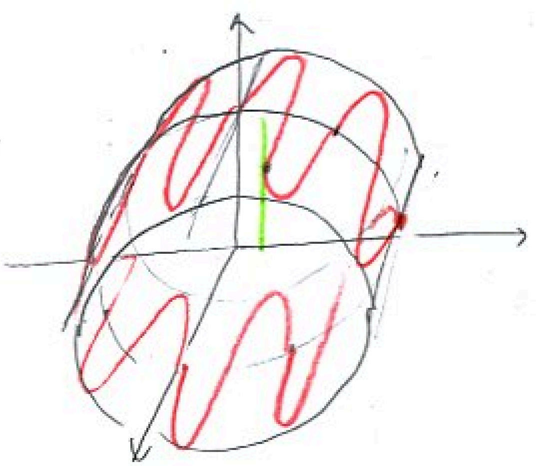
paraboloid
centered on
negative x-axis



6. (22pts) The curve $\mathbf{r}(t) = \langle \sin(8t), 2\cos t, 2\sin t\rangle$ is given, $0 \le t \le 4\pi$.
a) Sketch the curve in the coordinate system.
b) Find parametric equations of the tangent line to this curve when $t = \pi/3$ and sketch the tangent line.
c) Set up the integral for the length of the curve. Simplify the function inside the integral as much as possible, but do not evaluate the integral.

a) rotation in yz-plane, radius 2

combined with rotation in x-direction



c) $l = \int_0^{4\pi} |\vec{r}\,'(t)|\, dt$

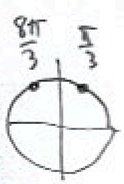
$= \int_0^{4\pi} \sqrt{(8\cos(8t))^2 + (-2\sin t)^2 + (2\cos t)^2}\, dt$

$= \int_0^{4\pi} \sqrt{64\cos^2(8t) + \underbrace{4\sin^2 t + 4\cos^2 t}_{4}}$

$= \int_0^{4\pi} 2\sqrt{16\cos^2(8t) + 1}\, dt$

b) $\vec{r}(t) = \langle \sin(8t), 2\cos t, 2\sin t\rangle$

$\vec{r}\,'(t) = \langle 8\cos(8t), -2\sin t, 2\cos t\rangle$



$\vec{r}\left(\frac{\pi}{3}\right) = \left\langle \frac{\sqrt{3}}{2}, 1, \sqrt{3} \right\rangle$

$\vec{r}\,'\left(\frac{\pi}{3}\right) = \langle -4, -\sqrt{3}, 1\rangle$

$x = \frac{\sqrt{3}}{2} - 4t$

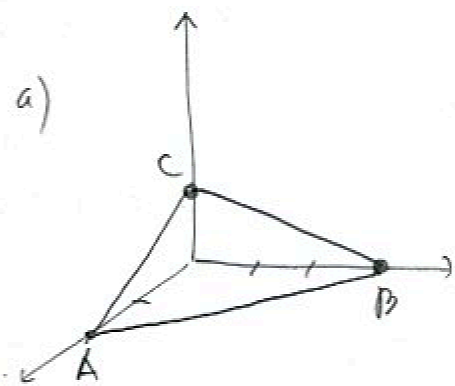
$y = 1 - \sqrt{3}t$

$z = \sqrt{3} + t$

7. (16pts) Consider the triangle whose vertices are intersections of the plane $\frac{x}{2}+\frac{y}{3}+\frac{z}{1}=1$ with the coordinate axes.

a) Draw the triangle.
b) Find the area of the triangle.
c) Is this a right triangle?

a)



x, y, z - intercepts are $2, 3, 1$

b) $\vec{AB} = \langle -2, 3, 0 \rangle$

$\vec{AC} = \langle -2, 0, 1 \rangle$

$\vec{BC} = \langle 0, -3, 1 \rangle$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 0 \\ -2 & 0 & 1 \end{vmatrix} = 3\vec{i} - (-2)\vec{j} + 6\vec{k} = 3\vec{i} + 2\vec{j} + 6\vec{k}$$

$$\text{area} = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right| = \frac{1}{2}\sqrt{3^2+2^2+6^2} = \frac{1}{2}\sqrt{49} = \frac{7}{2}$$

c) $\left.\begin{array}{l} \vec{AB}\cdot\vec{AC} = 4 \\ \vec{AC}\cdot\vec{BC} = 1 \\ \vec{AB}\cdot\vec{BC} = -9 \end{array}\right\}$ product of no two is 0, so none are perpendicular

Bonus (10pts) Standing at point $(2,1,1)$, you throw a rock with initial velocity vector $\mathbf{v_0} = \langle -1, 2, 8 \rangle$. Assuming gravity (let $g = 10$ here) acts in the usual negative z-direction, find the point where the rock hits the incline represented by the plane $3x + 4y + z - 5 = 0$.

$\vec{r}(0) = \langle 2, 1, 1 \rangle$

$\vec{v}(0) = \langle -1, 2, 8 \rangle$

$\vec{a}(t) = \langle 0, 0, -10 \rangle \mid \int dt$

$\vec{v}(t) = \langle 0, 0, -10t \rangle + \vec{C}$

$\underbrace{\langle -1, 2, 8 \rangle}_{\vec{v}(0)} = \langle 0, 0, 0 \rangle + \vec{C}$

so $\vec{C} = \langle -1, 2, 8 \rangle$

$\vec{v}(t) = \langle -1, 2, 8 - 10t \rangle$

$\vec{r}(t) = \langle -t, 2t, 8t - 5t^2 \rangle + \vec{D}$

$\underbrace{\langle 2, 1, 1 \rangle}_{\vec{r}(0)} = \langle 0, 0, 0 \rangle + \vec{D}$

so $\vec{D} = \langle 2, 1, 1 \rangle$

$\vec{r}(t) = \langle -t+2, 2t+1, -5t^2+8t+1 \rangle$

Rock is on plane when

$3(-t+2) + 4(2t+1) + (-5t^2+8t+1) - 5 = 0$

$-5t^2 + 13t + 6 = 0$

$5t^2 - 13t - 6 = 0$

$t = \frac{13 \pm \sqrt{169 - 4 \cdot 5 \cdot (-6)}}{2 \cdot 5} = \frac{13 \pm \sqrt{289}}{10} = \frac{13 \pm 17}{10} = 3, \cancel{-\frac{2}{5}}$

$\boxed{\begin{array}{l}\text{Hits plane at } \vec{r} \\ \vec{r}(3) = \langle -1, 7, -20 \rangle\end{array}}$