**Advanced Calculus 2 — Exam 1**
**MAT 526/626, Spring 2015 — D. Ivanšić**

**Name:** ____________________________

*Show all your work!*

*Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.*

**Theory 1.** (3pts) Define a uniformly continuous function.

**Theory 2.** (3pts) State the sequential criterion for continuity (only the affirmative version).

**Theory 3.** (3pts) State the Bolzano Intermediate Value Theorem.

## Type A problems (5pts each)

**A1.** Let $f : \mathbf{R} \to \mathbf{R}$ be such that $f(c) > 0$. Show that there exists a neighborhood $V_\delta(c)$ such that $f(x) > 0.9 \cdot f(c)$ for all $x \in V_\delta(c)$.

**A2.** Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and $A = \{x \in \mathbf{R} \mid f(x) \in [3, 5]\}$. If $(x_n)$ is a sequence such that $x_n \in A$ and $(x_n)$ converges to $c$, show that $c \in A$.

**A3.** Let $f : \mathbf{R} \to \mathbf{R}$ be a function. Show: $f(x)$ is continuous at $c$ if and only if the function $g(x) = f(x) + x$ is continuous at $c$. Don't do anything complicated.

**A4.** Let $f, g : \mathbf{R} \to \mathbf{R}$ be continuous functions such that $f(a) > g(a)$ and $f(b) < g(b)$, $a < b$. Show that there exists a $c \in (a, b)$ such that $f(c) = g(c)$.

**A5.** Let $f : [a, b] \to \mathbf{R}$ be a continuous function such that $f(x) < 0$, for all $x \in [a, b]$. Show there exists a number $d < 0$ such that $f(x) < d$, for all $x \in [a, b]$.

## Type B problems (8pts each)

**B1.** Let $f(x) = [[|x|]]$, where $[[x]]$ is the greatest integer operation (so, f is greatest integer of absolute value of $x$). Determine the numbers where the function is continuous and where it is not. Justify in detail (not just the picture).

**B2.** Give an example of a function $f$ that is discontinuous, but $f(x)^2$ is continuous. Is there an example where $f$ is discontinuous, but $f(x)^3$ is continuous? Why or why not?

**B3.** Let $f : [a, b] \to \mathbf{R}$ be continuous, and let $M$ be its maximum value. Show that $M = \sup\{f(x) \mid x \in [a, b] \cap \mathbf{Q}\}$.

**B4.** Show that the function $f(x) = \frac{1}{5x+3}$ is Lipschitz on the interval $[0, \infty)$.

**B5.** Let $f : [a, b] \to \mathbf{R}$ be a Lipschitz function such that $f(x) \neq 0$ for all $x \in [a, b]$. Show that the function $\frac{1}{f(x)}$ is Lipschitz, too.

**B6.** Let $f : [a, b] \to \mathbf{R}$ be continuous, and suppose $f$ takes on some value $V_1$ at least twice. Show that there is another function value $V_2 \neq V_1$ that is taken on at least twice.

## Type C problems (12pts each)

**C1.** Show that the function $f(x) = \sqrt{x}$ is uniformly continuous on its domain $[0, \infty)$.

**C2.** Show that the function $f(x) = \ln x$ is uniformly continuous on the interval $[1, \infty)$.

**C3.** Give an alternate proof that an increasing function $f : \mathbf{R} \to \mathbf{R}$ may have only countably many discontinuities: if $f$ is strictly increasing, let $D = \{c \mid f \text{ is not continuous at } c\}$ and define the function $g(x)$ as follows:

$$g(c) = \begin{cases} f(c), \text{ if } \lim_{x\to c-} f(x) = \lim_{x \to c+} f(x) \\ \text{any rational number in the interval } \left[\lim_{x\to c-} f(x), \lim_{x\to c+} f(x)\right], \text{ if } \lim_{x\to c-} f(x) < \lim_{x\to c+} f(x). \end{cases}$$

a) Show that $g$ is strictly increasing.
b) Show that $g(D) \subseteq \mathbf{Q}$.
c) Apply injectivity of $g$ to argue that $D$ is countable.
d) If $f$ is not strictly increasing, apply problem A3 to replace it by a strictly increasing function with the same discontinuity set.

| **Advanced Calculus 2 — Exam 2** <br> **MAT 526/626, Spring 2015 — D. Ivanšić** | **Name:**______________________ <br> *Show all your work!* |
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*Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.*

**Theory 1.** (3pts) Define the derivative of a function $f : I \to \mathbf{R}$, where $I$ is an interval.

**Theory 2.** (3pts) Define a convex function.

**Theory 3.** (3pts) State the Mean Value Theorem.

## Type A problems (5pts each)

**A1.** Is this function differentiable at 0: $f(x) = \begin{cases} x, & \text{if } x \in \mathbf{Q} \\ 0, & \text{if } x \notin \mathbf{Q}? \end{cases}$

**A2.** Find the limits: a) $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x^3}$ b) $\displaystyle\lim_{x\to 0+} \sqrt[n]{x}\ln x$.

**A3.** Let $f : \mathbf{R} \to \mathbf{R}$ be such that $f'''$ exists and $f'''(x) < 0$, and consider the Taylor polynomial $P_2$ for $f$ at $x_0$, whose graph is a parabola. Use Taylor's theorem to show that the graph of $f$ is above this parabola for $x < x_0$, and below it for $x > x_0$.

**A4.** Let $L(x)$ be a function such that $L'(x) = e^{x^2}$ (it exists, but cannot be written using elementary functions). Write expressions for the derivatives of the following functions:
a) $L(x)^2$ b) $L(L(x))$ c) $L(\sqrt{x})$

**A5.** Draw two pictures: the first illustrates how Newton's method works in a favorable setting, and the second shows how it can fail (that is, the next iteration is much farther from the solution than the current one).

**A6.** Suppose $f : \mathbf{R} \to \mathbf{R}$ is such that $f''$ exists and for some $a < c < b$ we have $f(a) = f(b) = 0$ and $f(c) > 0$. Show that there exists a point $d \in (a, b)$ such that $f''(d) < 0$ (use convexity).

## Type B problems (8pts each)

**B1.** Let $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$

Show that $f$ is differentiable at every point, find $f'(x)$, and show that $\displaystyle\lim_{x\to 0} f'(x)$ does not exist, hence $f'(x)$ is not continuous.

**B2.** Let $f : [a, \infty) \to \mathbf{R}$ be differentiable, $f(a) = b$ and suppose $m_1 \leq f'(x) \leq m_2$, for all $x \in (a, \infty)$. Use the Mean Value Theorem to show that the graph of $f$ must lie between the lines with slopes $m_1$ and $m_2$, passing through $(a, b)$. Conversely, does every smooth graph passing through $(a, b)$ between the two lines satisfy $m_1 \leq f'(x) \leq m_2$?

**B3.** Let $f(x) = \sqrt[3]{x}$.
a) Write the Taylor polynomial $P_3$ for $f$ at $x_0 = 8$.
b) Find the interval around 8 for which you can guarantee that $P_3$ approximates $f$ with accuracy $10^{-2}$.

**B4.** Use a Taylor polynomial to get a rational number that approximates $\sqrt{e}$ with accuracy $10^{-4}$.

**B5.** Show that the equation $x^3 + 2x^2 - 5 = 0$ has a solution and find an interval in which Newton's method converges, regardless of the starting point. Also, find how many iterations Newton's method will require to achieve accuracy $10^{-3}$.

**B6.** Find the limit: $\lim_{x \to 0+} (-\ln x)^{\ln(x+1)}$. (Note: for small $x > 0$, $\ln x < 0$, so we need a minus to ensure that the base is a positive number).

## Type C problems (12pts each)

**C1.** Let $I$ be an open interval and $f : I \to \mathbf{R}$. Suppose there exists a continuous function $g : I \to \mathbf{R}$ such that for every $u, v \in I$, there exists a $c$ between $u$ and $v$ such that

$$\frac{f(v) - f(u)}{v - u} = g(c).$$

Show that $f$ is differentiable on I and that $f'(x) = g(x)$, which implies that $f'$ is continuous.

**C2.** Let $f : (0, \infty) \to \mathbf{R}$ be differentiable on $(0, \infty)$. If the following statements are true, prove them, otherwise, find a counterexample.
a) If $\lim_{x \to \infty} f(x) = b$, then $\lim_{x \to \infty} f'(x) = 0$.
b) If $\lim_{x \to \infty} f'(x) = b$, where $b > 0$, then $\lim_{x \to \infty} f(x) = \infty$.
c) If $\lim_{x \to \infty} f'(x) = 0$, then $\lim_{x \to \infty} f(x)$ exists, and is a real number.

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| **Advanced Calculus 2 — Exam 3** <br> **MAT 526/626, Spring 2015 — D. Ivanšić** |

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*Show all your work!*

*Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.*

**Theory 1.** (3pts) Assuming tagged partitions have been defined, define the Riemann integral of a function $f : [a, b] \to \mathbf{R}$.

**Theory 2.** (3pts) State the second form of the Fundamental Theorem of Calculus (the one dealing with differentiability at a point).

**Theory 3.** (3pts) Explain how the Simpson rule is computed (formula) and what it represents geometrically.

## Type A problems (5pts each)

**A1.** One integral below can be evaluated using the substition theorem, and the other cannot. Evaluate the one that can, and explain why the the substitution theorem cannot be used on the other.

a) $\displaystyle\int_{\ln(\pi/4)}^{\ln(\pi/2)} e^x \sin e^x \, dx$  b) $\displaystyle\int_0^1 \frac{\ln x}{x} \, dx$

**A2.** If $F(x) = \displaystyle\int_{\sin x}^{\ln x} \frac{1}{1+t^8} \, dt$, find the expression for $F'(x)$.

**A3.** Let $f : [0, 1] \to \mathbf{R}$ be the function at right. Use Cauchy's criterion to show $f$ is not Riemann integrable.

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q} \\ -1, & \text{if } x \notin \mathbf{Q} \end{cases}$$

**A4.** If $f : [0, 1] \to \mathbf{R}$ is continuous and has the property $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f(x) = 0$ for all $x \in [0, 1]$.

**A5.** Show: if $f$ is Riemann integrable on $[a, b]$ and $\dot{\mathcal{P}}_n$ is a sequence of tagged partitions of $[a, b]$ such that $||\dot{\mathcal{P}}_n|| \to 0$, then $S(f, \dot{\mathcal{P}}_n) \to \int_a^b f$.

**A6.** Write the specific expression (i.e., with numbers, not variables) for the midpoint estimate $M_4$ of the integral $\int_1^3 \frac{1}{x} \, dx$, but do not evaluate it. Determine its accuracy.

## Type B problems (8pts each)

**B1.** Let $f : [1, \infty] \to \mathbf{R}$ be the function at right and let $F : [1, \infty) \to \mathbf{R}$, $F(x) = \int_1^x f$.
a) Calculate $F(x)$.
b) Draw the graphs of $f$ and $F$.
c) Where is $F$ continuous? Differentiable?

$$f(x) = \begin{cases} x, & \text{if } x \in [1, 2] \\ 1, & \text{if } x \in (2, 3] \\ 4 - x, & \text{if } (3, \infty) \end{cases}$$

**B2.** Let $f : [1,5] \to \mathbf{R}$ be the function at right.
a) Guess the value of $\int_1^5 f$.
b) Prove by definition of the Riemann integral that $\int_1^5 f$ is the number you guessed.

$$f(x) = \begin{cases} -3, & \text{if } x \in [1,2] \\ 4, & \text{if } x \in (2,5] \end{cases}$$

**B3.** Let $f : [0,1] \to \mathbf{R}$ be the function at right. Use the squeeze theorem to show $f$ is Riemann integrable on $[0,1]$.

$$f(x) = \begin{cases} \frac{n-1}{n}, & \text{if } x \in \left[\frac{1}{n}, \frac{1}{n-1}\right), n \geq 2 \\ 1, & \text{if } x = 0 \\ \frac{1}{2}, & \text{if } x = 1 \end{cases}$$

**B4.** Let $f : [a,b] \to \mathbf{R}$ be a function with the property: given any $\epsilon > 0$, there exists a step function $\phi : [a,b] \to \mathbf{R}$ such that $\phi$ uniformly approximates $f$ with accuracy $\epsilon$ on the interval $[a,b]$ (this means that $|f(x) - \phi(x)| < \epsilon$, for all $x \in [a,b]$). Show that $f$ is Riemann integrable.

**B5.** Let $f \in \mathcal{R}[-a,a]$, and let $f$ be an even function (note that $f$ need not be continous, so you may not use the substitution theorem). Show that $\int_{-a}^0 f = \int_0^a f$, and conclude that $\int_{-a}^a f = 2\int_0^a f$. (Hint: let $\dot{\mathcal{P}}_n$ be a sequence of tagged partitions of $[0,a]$ such that $||\dot{\mathcal{P}}_n|| \to 0$. Construct tagged partitions $\dot{\mathcal{Q}}_n$ of $[-a,0]$ such that $S(f,\dot{\mathcal{P}}_n) = S(f,\dot{\mathcal{Q}}_n)$, and use them to show integrals are same.)

**B6.** For the integral $\int_3^5 x^2 \sin x \, dx$, how many subintervals are needed so that the trapezoid estimate $T_n$ has accuracy $10^{-3}$?

## Type C problems (12pts each)

**C1.** If $p$ is a polynomial of degree at most 3, show the Simpson approximation $S_n$ is exact as follows, without using the error estimate. First note that it is enough to show this for the case of two subintervals (i.e., for $S_2$).
a) For each of the functions $f(x) = 1, x, x^2, x^3$ show that

$$\int_{a-h}^{a+h} f(x)\,dx = \frac{1}{3}h(f(a-h) + 4f(a) + f(a+h)).$$

b) Conclude that if $p$ is any polynomial of degree at most 3, $x_0 < x_1 < x_2$, $x_1 = \frac{x_0+x_2}{2}$, and $h = x_1 - x_0$, then

$$\int_{x_0}^{x_2} p(x)\,dx = \frac{1}{3}h(p(x_0) + 4p(x_1) + p(x_2)),$$

which proves the Simpson approximation is exact for $S_2$.
c) Conclude the Simpson approximation $S_n$ is exact for any polynomial $p$ of degree at most 3, and any even $n$.