Spring 2010: MAT 526, Exam 1

Do all the theory problems. Then do at least five problems, one of which is of a different type than others (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define the derivative of f at a point a.

Theory 2. (3pts) State the Mean Value Theorem.

Theory 3. (3pts) Define a convex function and state the theorem on convex functions.

Type A problems (5pts each)

A1. Let f be differentiable on an open interval around a. Interpret with a picture and show that

$$\lim_{h\to 0} \frac{f(a+3h)-f(a-2h)}{5h} = f'(a).$$

A2. Show that the function $f(x) = \sin\frac{1}{x}$ is not uniformly continuous on $(0,1]$.

A3. Show that $f(x) = \sqrt[3]{x}$ is not a Lipschitz function on the interval $[0,2]$.

A4. Let f have a continuous derivative on a closed interval $[a,b]$. Show that f is Lipschitz on this interval.

Type B problems (8pts each)

B1. Show that, for any $n \in \mathbf{N}$, $\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x^n} = 0$. *(Hint: requires a trick.)*

B2. State and prove the Uniform Continuity Theorem.

B3. Let $f(x) = \cos x$.
a) Write the 4th Taylor polynomial for f at 0.
b) Use the error estimate to find an interval on which P_4 approximates f with accuracy 10^{-4}.
c) Now note that $P_4 = P_5$. Use the error estimate for P_5 to find an interval on which P_4 approximates f with accuracy 10^{-4}. Did you get a bigger interval than in b)?

B4. Let $f : [-1,4] \to \mathbf{R}$ be continuous on $[-1,4]$ and differentiable on $(-1,4)$, and suppose that $-2 \le f'(x) \le 7$ for all $x \in (-1,4)$. If $f(-1) = 21$, establish the range of values that $f(4)$ can take. Give examples to show that the upper and lower bound for $f(4)$ can be achieved.

B5. Let $f : I \to \mathbf{R}$, where I is an open interval, be differentiable on I and let $a \in I$. Show: if $\lim_{x\to a} f'(x)$ exists, it is equal to $f'(a)$.

B6. Use Newton's method to find $\sqrt[5]{40}$ with accuracy 10^{-6}. Your error estimate must assure that $|x_n - r| < 10^{-6}$.

Type C problems (12pts each)

C1. Let $f(x) = e^{-\frac{1}{x}}$, if $x > 0$, and $f(x) = 0$, if $x \leq 0$.
a) Use induction to show that $f^{(n)}(x) = \dfrac{Q_n(x)}{x^{2n}} e^{-\frac{1}{x}}$ for $x > 0$, where $Q_n(x)$ is a polynomial of degree $n-1$ with the property $Q_n(0) = (-1)^n$.
b) Use a) and the statement of **B1** to show $f^{(n)}(0) = 0$ for every $n \in \mathbf{N}$.
c) Use b) to write the n-th Taylor polynomial $P_n(x)$ for f at 0. By simply comparing $f(x)$ and $P_n(x)$ show that the remainder $R_n(x)$ does not converge to 0 as $n \to \infty$ for any $x > 0$ (no estimates needed here of R_n).

Spring 2010: MAT 526, Exam 2

Do all the theory problems. Then do at least five problems, one of which is of a different type than others (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Assuming tagged partitions have been defined, define the Riemann integral of a function $f : [a, b] \to \mathbf{R}$.

Theory 2. (3pts) State the squeeze theorem on Riemann-integrable functions.

Theory 3. (3pts) Explain how the trapezoid rule is computed and what it represents geometrically.

Type A problems (5pts each)

A5. Give an example of a function that is not Riemann-integrable, and show, using, the definition, that it is not Riemann-integrable.

A6. If $E \subset [a, b]$ is a finite set and $f : [a, b] \to \mathbf{R}$ is such that $f(x) = 0$ for $x \in [a, b] - E$, prove by definition that $\int_a^b f = 0$.

A7. Prove the Mean Value Theorem for Integrals: if $f(x)$ is continuous on $[a, b]$, there exists a $c \in [a, b]$ such that $\int_a^b f = f(c)(b - a)$.

A8. Let $f(x) = n$, if $x \in [n, n + 1)$, where $n \in \mathbf{Z}$, and $f(x) = 0$ if $x < -2$ or $x \geq 2$. Draw the graph of $F : \mathbf{R} \to \mathbf{R}$, if $F(x) = \int_{-1}^x f$. Where is F continuous? Differentiable?

A9. If $f : [0, 1] \to \mathbf{R}$ is continuous and has the property $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f(x) = 0$ for all $x \in [0, 1]$.

A10. If $F(x) = \int_{x^2}^{e^x} \sin t \, dt$, find $F'(1)$.

Type B problems (8pts each)

B7. Let $f(x) : [a, b] \to \mathbf{R}$ be such that $f(x) = k$ for every $x \in [c, d] \subseteq [a, b]$, and $f(x) = 0$ if $x \notin [c, d]$. Prove, using the definition of a Riemann integral, that $\int_a^b f = k(b - a)$.

B8. Show, using the definition of a Riemann integral: if f is Riemann-integrable on $[a, b]$ and $|f(x)| \leq M$ for all $x \in [a, b]$, then $|\int_a^b f| \leq M(b - a)$.

B9. State and prove the theorem on Riemann-integrability of continuous functions.

B10. State and prove the Fundamental Theorem of Calculus, First Form.

B11. Suppose $f(x)$ is continuous on $[a, b]$, $f(x) \geq 0$ for all $x \in [a, b]$ and there is a $c \in [a, b]$ such that $f(c) > 0$. Show that $\int_a^b f > 0$.

B12. Use the midpoint rule to estimate the integral $\int_0^1 e^{-x^2} dx$ to accuracy 10^{-4}.

Type C problems (12pts each)

C2. If p is a polynomial of degree at most 3, show that the Simpson Approximation is exact. Note that it is enough to show this for the case of two subintervals (i.e., for S_2).

Fall 2009: MAT 526, Exam 3

Do all the theory problems. Then do at least five problems, one of which is of a different type than others (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define when a sequence of functions $(f_n) : A \to \mathbf{R}$ converges uniformly on A_0 to a function $f : A_0 \to \mathbf{R}$.

Theory 2. (3pts) State the theorem on the interchange of limit and derivative.

Theory 3. (3pts) Assuming the exponential function is defined, define the natural logarithm function and the generalized power function x^α.

Type A problems (5pts each)

A17. Let $f_n : \mathbf{R} \to \mathbf{R}$ be defined by $f_n(x) = \frac{1}{1+nx^2}$. Find $\lim f_n$. Is the convergence uniform?

A18. Let C, $S : \mathbf{R} \to \mathbf{R}$ be any two functions with the property $S'(x) = C(x)$ and $C'(x) = -S(x)$. Show that $C(x)^2 + S(x)^2$ is constant.

A19. Let $||f||$ denote the uniform norm of a bounded function for a function $f : A \to \mathbf{R}$. Show that for any two bounded functions f, $g : A \to \mathbf{R}$ we have the triangle inequality: $||f + g|| \leq ||f|| + ||g||$.

A20. Give an example of a sequence of continuous functions $(f_n) : [0, 1] \to \mathbf{R}$ that converge to a function $f : [0, 1] \to \mathbf{R}$, but $\lim \int_0^1 f_n(x)\, dx$ is not equal to $\int_0^1 f(x)\, dx$.

A21. Give an example of a sequence of functions $(f_n) : [0, 1] \to \mathbf{R}$, each of which is discontinuous at every point, that converge uniformly to a continuous function $f : [0, 1] \to \mathbf{R}$.

Type B problems (8pts each)

B19. Let $f_n : [0, \infty) \to \mathbf{R}$ be defined by $f_n(x) = \frac{nx}{1+nx}$. Find $\lim f_n$. Show that (f_n) converges uniformly on every interval $[A, \infty)$, $A > 0$, but does not converge uniformly on $[0, \infty)$.

B20. State and prove the theorem on the interchange of limit and continuity.

B21. A function $g : \mathbf{R} \to \mathbf{R}$ has the property that there exist $m \in \mathbf{N}$ and $\alpha \in \mathbf{R}$ such that $g(0) = g'(0) = \cdots = g^{(m-1)}(0) = 0$ and $g^{(m)}(x) = \alpha^m g(x)$. Show that $g(x) = 0$ for all $x \in \mathbf{R}$.

B22. Let C, $S : \mathbf{R} \to \mathbf{R}$ be the two functions whose existence we established. Show directly, without using addition formulas, that $S(2x) = 2S(x)C(x)$. *(Hint: what do we know about functions f for which $f'' = -f$?)*

B23. In a proof in class, for functions E_n used to define the exponential function, we showed that for $m > n$, $|E_m(x) - E_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}\left(1 + \frac{|x|}{n} + \left(\frac{|x|}{n}\right)^2 + \dots \left(\frac{|x|}{n}\right)^{m-n-1}\right)$. Use this inequality to help you calculate e to 6 decimal places.

Type C problems (12pts each)

C4. Let $(f_n) : (0, \infty) \to \mathbf{R}$ be defined by $f_n(x) = \dfrac{\sin \frac{x}{n}}{\frac{x}{n}}$. Find $\lim f_n$ and show that the convergence is uniform on interval $(0, A]$ for every $A > 0$, but it is not uniform on $(0, \infty)$.