Calculus 1 — Exam 5
MAT 250, Spring 2015 — D. Ivanšić

Name: Saul Ocean

Show all your work!

Find the following antiderivatives.

1. (3pts) $\int \frac{1}{\sqrt[3]{x^2}}\,dx = \int x^{-\frac{2}{3}}\,dx = \frac{x^{\frac{1}{3}}}{\frac{1}{3}} = 3x^{1/3} + C = 3\sqrt[3]{x} + C$

2. (3pts) $\int \frac{5}{\sqrt{1-x^2}}\,dx = 5\arcsin x + C$

3. (3pts) $\int e^{3x+7}\,dx = \frac{e^{3x+7}}{3} + C$

4. (7pts) $\int \frac{u^2-u+1}{\sqrt{u}}\,du = \int \frac{u^2-u+1}{u^{1/2}}\,du = \int u^{\frac{3}{2}} - u^{\frac{1}{2}} + u^{-\frac{1}{2}}\,du$

$= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + C$

5. (7pts) Find $f(x)$ if $f'(x) = \cos(3x) + \sec^2 x$ and $f(0) = 4$.

$f(x) = \frac{\sin(3x)}{3} + \tan x + C$

$4 = f(0) = \frac{\sin 0}{3} + \tan 0 + C$

$4 = 0 + 0 + C, \quad C = 4$

$f(x) = \frac{\sin(3x)}{3} + \tan x + 4$

6. (8pts) Find $f(x)$ if $f''(x) = \frac{4}{x^3}$, $f'(1) = 3$ and $f(2) = -2$.

$f''(x) = 4x^{-3}$

$f'(x) = 4\frac{x^{-2}}{-2} + C = -2x^{-2} + C$

$3 = f'(1) = -2 \cdot 1 + C$

$C = 5$

$f'(x) = -2x^{-2} + 5$

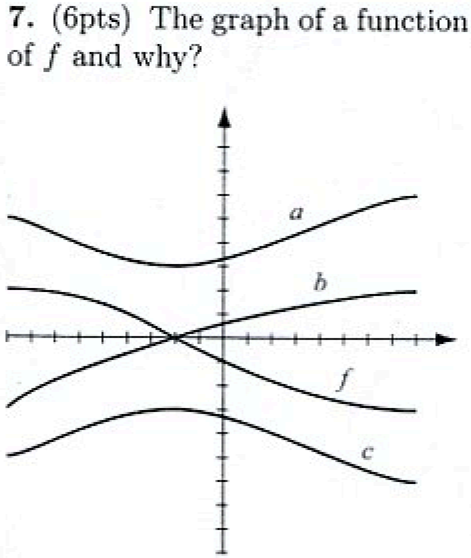
$f(x) = -2\frac{x^{-1}}{-1} + 5x + D$

$= 2x^{-1} + 5x + D$

$-2 = f(2) = 2 \cdot \frac{1}{2} + 10 + D, \quad D = -13$

$f(x) = \frac{2}{x} + 5x - 13$

7. (6pts) The graph of a function f is shown. Which of the other graphs is an antiderivative of f and why?



Where $f(x)=0$, antider. has horizontal tangent line, which only a and c have.

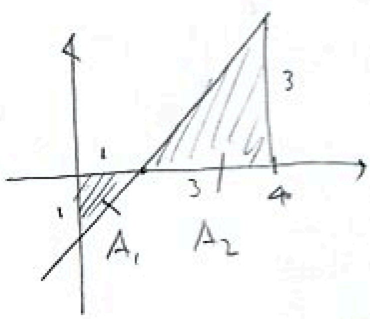
Where $f>0$, antiderivative is increasing

Where $f<0$, — is decreasing

Graph of $\boxed{c}$ satisfies these requirements

8. (15pts) Find $\int_0^4 x - 1\,dx$ in two ways (they'd better give you the same answer!):
a) Using the "area" interpretation of the integral. Draw a picture and use area of triangles.
b) Using the Evaluation Theorem.

a)



$$\int_0^4 x-1\,dx = -A_1 + A_2 = -\frac{1}{2}\cdot 1\cdot 1 + \frac{1}{2}\cdot 3\cdot 3$$

$$= -\frac{1}{2} + \frac{9}{2} = 4$$

same!

b) $$\int_0^4 x-1\,dx = \left(\frac{x^2}{2} - x\right)\Big|_0^4 = \frac{4^2}{2} - 4 - (0-0) = 8-4 = 4$$

Use the substitution rule in the following integrals:

9. (8pts) $\int (3x^2 - 2x)\sqrt{x^3 - x^2 + 1}\, dx = \begin{bmatrix} u = x^3 - x^2 + 1 \\ du = (3x^2 - 2x)dx \end{bmatrix} = \int \sqrt{u}\, du$

$$= \int u^{\frac{1}{2}} du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}(x^3 - x^2 + 1)^{\frac{3}{2}} + C$$

10. (10pts) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{2 + \cos x}\, dx = \begin{bmatrix} u = 2 + \cos x & x = \frac{\pi}{2},\ u = 2 \\ du = -\sin x\, dx & x = 0,\ u = 3 \\ -du = \sin x\, dx & \end{bmatrix} = \int_3^2 \frac{-du}{u}$

$$= \int_2^3 \frac{1}{u}\, du = \ln|u|\Big|_2^3 = \ln 3 - \ln 2$$

11. (10pts) $\int_3^5 \frac{e^{\frac{1}{x}}}{x^2}\, dx = \begin{bmatrix} u = \frac{1}{x} & x = 5,\ u = \frac{1}{5} \\ du = -\frac{1}{x^2}\, dx & x = 3,\ u = \frac{1}{3} \\ -du = \frac{1}{x^2}\, dx & \end{bmatrix} = \int_{\frac{1}{3}}^{\frac{1}{5}} e^u(-du) = \int_{1/5}^{1/3} e^u\, du$

$$= e^u\Big|_{1/5}^{1/3} = e^{1/3} - e^{1/5} = \sqrt[3]{e} - \sqrt[5]{e}$$

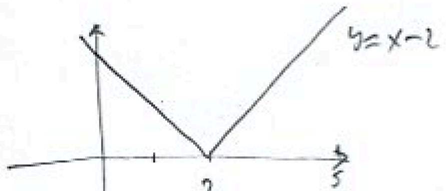
12. (10pts) Evaluate the following integral by breaking it up into two integrals without absolute value and evaluating each one. The graph of $y = |x - 2|$ might help.

$$\int_1^5 |x-2|\, dx = \int_1^2 |x-2|\, dx + \int_2^5 |x-2|\, dx = \int_1^2 2 - x\, dx + \int_2^5 x - 2\, dx$$

$$= \left(2x - \frac{x^2}{2}\right)\Big|_1^2 + \left(\frac{x^2}{2} - 2x\right)\Big|_2^5$$

$$= 2(2-1) - \frac{1}{2}(2^2 - 1^2) + \frac{1}{2}(5^2 - 2^2) - 2(5-2)$$

$$= 2 - \frac{3}{2} + \frac{21}{2} - 6 = 5$$



$$|x-2| = \begin{cases} x - 2 & \text{if } x \geq 2 \\ 2 - x & \text{if } x < 2 \end{cases}$$

13. (10pts) The rate at which water is flowing into a tank is $-t^2 + 10t - 9$ liters per minute.
a) Use the Net Change Theorem to find by how much the volume of water in the tank has changed from $t = 0$ to $t = 6$.
b) If at time $t = 0$ there were 23 liters of water in the tank, how many were there at time $t = 6$?

a) $V'(t) = -t^2 + 10t - 9$

$$V(6) - V(0) = \int_0^6 V'(t)\,dt = \int_0^6 (-t^2 + 10t - 9\ dt = \left(-\frac{t^3}{3} + 5t^2 - 9t\right)\Big|_0^6$$

$$= \left(-\frac{6^3}{3} + 5\cdot 6^2 - 9\cdot 6\right) - 0 = -\frac{6\cdot 6\cdot \cancel{6}^{\,2}}{\cancel{3}_{\,1}} + 5\cdot 36 - 54 =$$

$$= -72 + 180 - 54 = 180 - 126 = 54 \text{ liters}$$

b) $V(6) = V(0) + \left(\begin{matrix}\text{chg. in vol.}\\ \text{from } 0 \text{ to } 6\end{matrix}\right) = 23 + 54 = 77$ liters

Bonus. (10pts) A rocket takes off vertically from the ground, accelerating at constant acceleration. If at time $t = 10$ seconds it is at height 900 meters, what was its acceleration?

$s''(t) = a$

$s'(t) = at + C$

$0 = s'(0) = a\cdot 0 + C$ so $C = 0$

$s'(t) = at$

$s(t) = a\frac{t^2}{2} + D$

$0 = s(0) = 0 + D$ so $D = 0$

$900 = s(10) = a\frac{10^2}{2}$

$900 = \frac{100a}{2}$

$a = \frac{2\cdot 9\cancel{00}}{1\cancel{00}} = 18$

Acceleration is $18\ m/s^2$