Calculus 1 — Exam 4
MAT 250, Spring 2015 — D. Ivanšić

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Show all your work!

1. (30pts) Let $f(x) = \dfrac{x^2}{x^2+1}$. Draw an accurate graph of f by following the guidelines.
a) Find the intervals of increase and decrease, and local extremes.
b) Find the intervals of concavity and points of inflection.
c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
d) Use information from a)–d) to sketch the graph.

$$f'(x) = \frac{2x(x^2+1) - x^2\cdot 2x}{(x^2+1)^2} = \frac{2x^3+2x-2x^3}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$

$$f''(x) = \frac{2\cdot(x^2+1)^2 - 2x\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4} = \frac{2(x^2+1)(x^2+1-4x^2)}{(x^2+1)^3}$$

$$= \frac{2(1-3x^2)}{(x^2+1)^3}$$

a) critical pts; $2x=0$, $x=0$ ($x^2+1>0$ so $f'(x)$ is always defined)

$(x^2+1)^2>0$ so f' has same sign as x

		0	
f'	$-$	0	$+$
f	$\searrow$	loc. min.	$\nearrow$

b) 2nd order crit. pts.

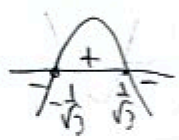
$1-3x^2=0$ $\quad x = \pm\frac{1}{\sqrt{3}}$ $\quad$ ($x^2+1>0$ so f'' is always defined)

$3x^2=1$

$x^2=\frac{1}{3}$

		$-\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$	
f''	$-$	0	$+$	0	$-$
f	CD	IP	CU	IP	CD

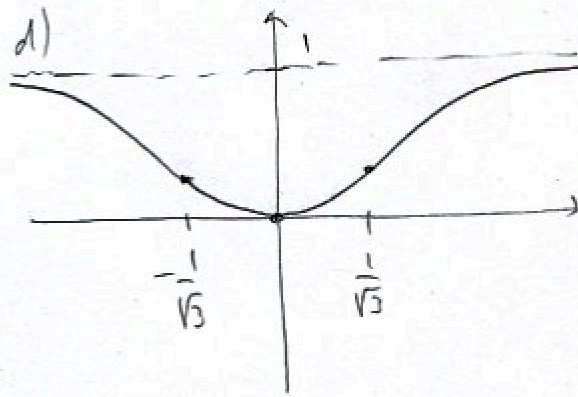
Since $x^2+1>0$
f has sign of $1-3x^2$



c) $\lim\limits_{x\to\infty} \dfrac{x^2}{x^2+1} = \lim\limits_{x\to\infty} \dfrac{x^2}{x^2(1+\frac{1}{x^2})}$

$= \dfrac{1}{1+0} = 1$ $\quad$ Same for $x\to-\infty$

d)



x	$\frac{x^2}{x^2+1}$
0	0
$\frac{1}{\sqrt{3}}$	$\frac{\frac{1}{3}}{\frac{1}{3}+1}\cdot\frac{3}{3} = \frac{1}{4}$
$-\frac{1}{\sqrt{3}}$	$\frac{1}{4}$

2. (14pts) Let $f(x) = 24x^{\frac{1}{2}} - 2x^{\frac{3}{2}}$. Find the absolute minimum and maximum values of f on the interval $[1, 9]$.

$$f'(x) = 24 \cdot \frac{1}{2} x^{-\frac{1}{2}} - 2 \cdot \frac{3}{2} x^{\frac{1}{2}}$$

$$= 12x^{-\frac{1}{2}} - 3x^{\frac{1}{2}}$$

$$= 3x^{-\frac{1}{2}}(4-x) = \frac{3(4-x)}{\sqrt{x}}$$

Crit pts: $f' = 0$ or f' dne

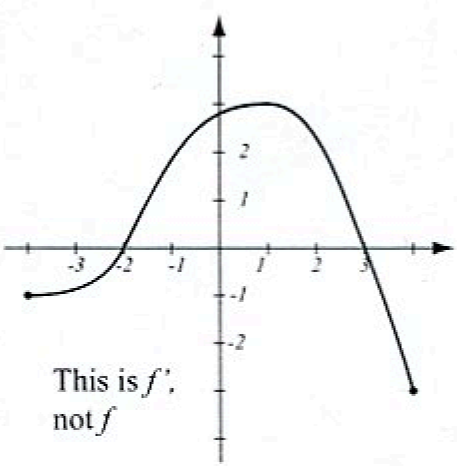
$4 - x = 0$ $x = 0$

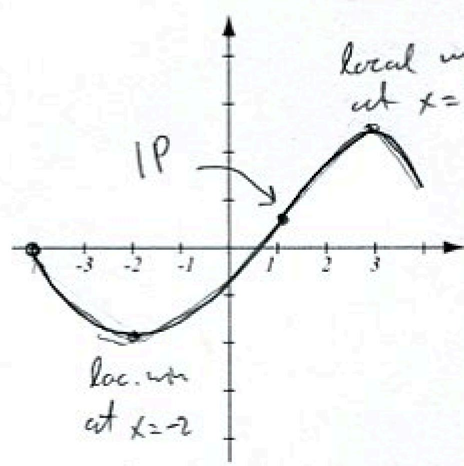
$x = 4$ (not in $[1,9]$

x	$24x^{\frac{1}{2}} - 2x^{\frac{3}{2}}$	
4	$24 \cdot 2 - 2 \cdot 8 = 32$	abs max
1	$24 - 2 = 22$	
9	$24 \cdot 3 - 2 \cdot 27 = 18$	abs min

$72 - 54$

3. (16pts) Let f be continuous on $[-4, 4]$. The graph of its derivative f' is drawn below. Use the graph to answer (sign charts may help):
a) What are the intervals of increase and decrease of f? Where does f have a local minimum or maximum?
b) What are the intervals of concavity of f? Where does f have inflection points?
c) Use the information gathered in a) and b) to sketch the graph of f at right, if $f(-4) = 0$.





a) increasing when $f' > 0$: $(-2, 3)$ loc. min at $x = -2$
decreasing when $f' < 0$: $(-4, -2) \cup (3, 4)$ max at $x = 3$

b) CU when f' is increasing: $(-4, 1)$ infl. pt. at $x = 1$
CD when f' is decreasing: $(1, 4)$

4. (16pts) Let $f(x) = \sin^2 x$, $0 \le x \le 2\pi$. Find the intervals of concavity and points of inflection for f.

$f'(x) = 2\sin x \cos x$

$f''(x) = 2(\cos x \cos x + \sin x(-\sin x))$

$= 2(\cos^2 x - \sin^2 x)$

2nd order critical pts:

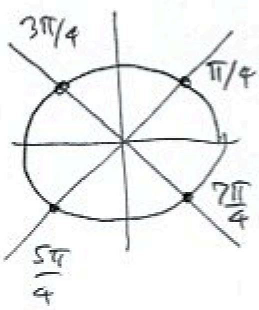
$f'' = 0$ $\qquad$ f'' dne (always defined)

$\cos^2 x - \sin^2 x = 0$

$\cos^2 x = \sin^2 x$

$\cos x = \pm \sin x$

(x coord $= \pm$ y-coord)

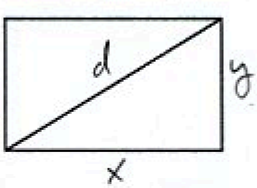


	0		$\frac{\pi}{4}$		$\frac{3\pi}{4}$		$\frac{5\pi}{4}$		$\frac{7\pi}{4}$		2π
f''		$+$	0	$-$	0	$+$	0	$-$	0	$+$	
f		CU	$\bullet$	CD	$\bullet$	CU	$\bullet$	CD	$\bullet$	CU	

$\bullet$ = inflection pts at

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

5. (24pts) Among all rectangles of area 100 square meters, find the one which has the shortest diagonal.



$x \cdot y = 100 \Rightarrow y = \frac{100}{x}$

$d^2 = x^2 + y^2 = x^2 + \left(\frac{100}{x}\right)^2 = x^2 + \frac{10^4}{x^2}$ $\quad$ (Maximizing d^2 is same as maximizing d)

Job: Maximize $f(x) = x^2 + \frac{10^4}{x^2}$ on $(0, \infty)$ $\quad$ ($x=0$ gives 0 area, so not considered)

$f'(x) = 2x + 10^4(-2x^{-3}) = 2x - \frac{2 \cdot 10^4}{x^3}$

$2x - \frac{2 \cdot 10^4}{x^3} = 0 \quad | \cdot \frac{x^3}{2}$

$x^4 - 10^4 = 0$

$x^4 = 10^4$

$x = \pm 10$ $\quad$ (-10 not in domain)

$f''(x) = 2 + 10^4 \cdot 6x^{-4}$

$f''(10) > 0$ so f has a local min at $x = 0$.

Since it is the only critical pt. on an open interval, f has an absolute max there, too.

Bonus. (10pts) Suppose $f(x) > 0$ and f is concave up. Let $g(x) = (f(x))^2$.
a) Find the expression for $g''(x)$.
b) Show that g is concave up.

a) $g'(x) = 2 f(x) f'(x)$

$g''(x) = 2(f'(x) f'(x) + f(x) \cdot f''(x))$

$= 2((f'(x))^2 + f(x) \cdot f''(x))$

b) Since f is concave up, $f''(x) > 0$. Additionally, $f(x) > 0$ and $(f'(x))^2 \geq 0$ so $g''(x) > 0$, which means g is concave up.