Calculus 1 — Exam 3 MAT 250, Spring 2015 — D. Ivanšić	Name: Saul Ocean

Show all your work!

Differentiate and simplify where appropriate:

1. (3pts) $\frac{d}{dx} e^{x^2+3x-1} = e^{x^2+3x-1}\cdot(2x+3)$

2. (4pts) $\frac{d}{dx}\ln(\tan^2 x) = \frac{d}{dx}\, 2\ln(\tan x) = 2\frac{1}{\tan x}\cdot\sec^2 x = 2\frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x} = \frac{2}{\sin x\cos x}$

or $= \frac{2\sec^2 x}{\tan x}$

3. (6pts) $\frac{d}{dt}\frac{t^2-3t}{7^t} = \frac{(2t-3)7^t-(t^2-3t)7^t\cdot\ln 7}{(7^t)^2} = \frac{7^t(2t-3-\ln 7\, t^2+3\ln 7\, t)}{(7^t)^2}$

$= \frac{-\ln 7\, t^2+(3\ln 7+2)t-3}{7^t}$

4. (7pts) $\frac{d}{dx}\ln\frac{\sin x+\cos x}{\sin x-\cos x} = \frac{d}{dx}\left(\ln(\sin x+\cos x)-\ln(\sin x-\cos x)\right)$

$= \frac{1}{\sin x+\cos x}(\cos x-\sin x) - \frac{1}{\sin x-\cos x}(\cos x+\sin x) = \frac{-(\cos x-\sin x)^2-(\cos x+\sin x)^2}{(\sin x+\cos x)(\sin x-\cos x)}$

$= \frac{-(\cos^2 x-2\sin x\cos x+\sin^2 x)-(\cos^2 x+2\sin x\cos x+\sin^2 x)}{\sin^2 x-\cos^2 x} = \frac{-2}{\sin^2 x-\cos^2 x}$

$= \frac{2}{\cos^2 x-\sin^2 x}$

5. (8pts) $\frac{d}{du}\left(u\arctan u-\frac{1}{2}\ln(1+u^2)\right) =$

$= 1\cdot\arctan u+\underbrace{u\frac{1}{1+u^2}-\frac{1}{2}\frac{1}{1+u^2}\cdot 2u}_{\text{cancel}}$

$= \arctan u$

6. (10pts) Use logarithmic differentiation to find the derivative of $y = x^{\sqrt{x}}$.

$y = x^{\sqrt{x}}$

$\ln y = \ln x^{\sqrt{x}}$

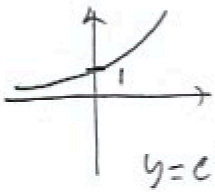
$\ln y = \sqrt{x}\ln x \quad \Big| \frac{d}{dx}$

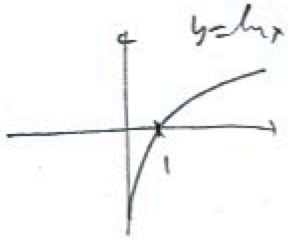
$\frac{1}{y}y' = \frac{1}{2\sqrt{x}}\ln x+\sqrt{x}\cdot\frac{1}{x}$

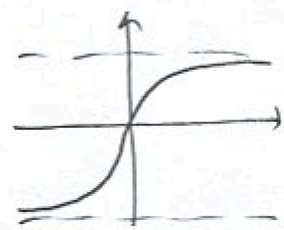
$y' = y\left(\frac{1}{2\sqrt{x}}\ln x+\frac{1}{\sqrt{x}}\right)$

$y' = x^{\sqrt{x}}\,\frac{\ln x+2}{2\sqrt{x}}$

7. (4pts) Draw the graphs of e^x, $\ln x$ and $\arctan x$ (each in its coordinate system).

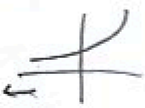






Find the limits algebraically. Graphs of basic functions will help, as will L'Hospital's rule, where appropriate.

8. (2pts) $\lim_{x\to-\infty} 5^x = 0$



9. (6pts) $\lim_{x\to0^+} \arctan\left(4 - \frac{1}{x}\right) = \arctan\left(4 - \frac{1}{0+}\right) = \arctan(4-\infty) = \arctan(-\infty)$

$= -\frac{\pi}{2}$

10. (6pts) $\lim_{x\to0} \dfrac{\overset{\to 1-1=0}{1-\cos x}}{\underset{\to 0}{x^2}} \underset{L'H}{=} \lim_{x\to0} \dfrac{\overset{\to 0}{-(-\sin x)}}{\underset{\to 0}{2x}} \underset{L'H}{=} \lim_{x\to0} \dfrac{\cos x}{2} = \dfrac{\cos 0}{2} = \dfrac{1}{2}$

11. (6pts) $\lim_{x\to0^+} x^3 \ln x = \lim_{x\to0+} \dfrac{\overset{\to-\infty}{\ln x}}{\frac{1}{x^3}} \underset{L'H}{=} \lim_{x\to0+} \dfrac{\frac{1}{x}}{-3x^{-4}} = \lim_{x\to0+} -\dfrac{\frac{1}{x}\cdot x^4}{3}$

$0\cdot(-\infty)$ $\quad \to \frac{1}{0+} = \infty$ $\quad -\frac{3}{x^4}$

$= \lim_{x\to0+} -\frac{x^3}{3} = 0$

12. (10pts) $\lim_{x\to\infty} \underbrace{(x^2+3x-1)^{\frac{1}{x}}}_{y} = e^{\lim_{x\to\infty} \ln y} = e^{0} = 1$

$= \lim_{x\to\infty} \frac{1}{x}\cdot\frac{2}{1} = 0$

$\ln y = \ln(x^2+3x-1)^{\frac{1}{x}} = \frac{1}{x}\ln(x^2+3x-1)$

$\lim_{x\to\infty} \dfrac{\overset{\to \ln\infty=\infty}{\ln(x^2+3x-1)}}{\underset{\to\infty}{x}} = \lim_{x\to\infty} \dfrac{\frac{1}{x^2+3x-1}(2x-3)}{1} = \lim_{x\to\infty} \dfrac{2x-3}{x^2+3x-1} = \lim_{x\to\infty} \dfrac{\cancel{x}\left(2-\frac{3}{x}\right)}{\cancel{x^2}\left(1+\frac{3}{x}-\frac{1}{x^2}\right)}$

$\to 0$ $\quad x \quad \to 0 \quad \to 0$

13. (10pts) Let $f(x) = \sqrt[3]{x}$.
a) Write the linearization of $f(x)$ at $a = 8$.
b) Use the linearization to estimate $\sqrt[3]{8.3}$ and compare to the calculator value of 2.024694.

a) $f(x) = x^{\frac{1}{3}}$ $\qquad f(8) = 2$

$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$ $\qquad f'(8) = \frac{1}{3} \frac{1}{8^{\frac{2}{3}}} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

$$L(x) = 2 + \frac{1}{12}(x-8) = \frac{1}{12}x + 2 - \frac{\cancel{8}^{2}}{\cancel{12}_{3}} = \frac{1}{12}x + \frac{4}{3}$$

$$b)\ L(8.3) = 2 + \frac{1}{12}(8.3-8) = 2 + \frac{1}{\cancel{12}_{4}} \cdot \cancel{0.3}^{0.1} = 2 + \frac{0.1}{4} = 2 + \frac{1}{40} = 2 + 0.025$$

$= 2.025$, pretty close to 2.024694

14. (10pts) Radius of a sphere r is measured to be 10 meters, with maximum error 5 centimeters. Use differentials to estimate the maximum possible error, the relative error and the percentage error when computing the surface area A of the sphere ($A = 4\pi r^2$, leave your answer in terms of π).

$A = 4\pi r^2 \qquad A' = 8\pi r$

$\Delta A \approx dA = A' \cdot dr = 8\pi r\, dr$

$\Delta A \approx dA = 8\pi \cdot 10 \cdot 0.05 = 4\pi$

relative error $= \dfrac{\cancel{4\pi}}{\cancel{4\pi} \cdot 10^2} = \dfrac{1}{100}$ $\qquad$ Percentage error $= 1\%$

15. (8pts) Let $f(x) = e^x + 3x + 4$. Use the theorem on derivatives of inverses to find $(f^{-1})'(5)$.

$f(x) = e^x + 3x + 4$

$f'(x) = e^x + 3$

Need $f^{-1}(5)$, which is solution of

$f(x) = 5$

$e^x + 3x + 4 = 5$

$e^x + 3x = 1$

guess: $x = 0 = f^{-1}(5)$

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(0)} = \frac{1}{e^0 + 3} = \frac{1}{4}$$

Bonus. (10pts) Find the limit. (Note: for small $x > 0$, $\ln x < 0$, so we need a minus to ensure that the base is a positive number).

$$\lim_{x \to 0+} \underbrace{(-\ln x)^{\ln(x+1)}}_{y} = e^{\lim_{x\to 0+} \ln y} = e^0 = 1$$

$$\ln y = \ln\left(\left(-\ln x\right)^{\ln(x+1)}\right) = \ln(x+1)\ln(-\ln x)$$

$$\lim_{x\to 0+} \ln(x+1)\ln(-\ln x) = \lim_{x\to 0+} \frac{\overset{\to \ln\infty = \infty}{\ln(-\ln x)}}{\underset{\to \frac{1}{\ln 1+} = \frac{1}{0+} = \infty}{\frac{1}{\ln(x+1)}}} \underset{L'H}{=} \lim_{x\to 0+} \frac{\frac{1}{-\ln x}\cdot\left(-\frac{1}{x}\right)}{-(\ln(x+1))^{-2}\cdot\frac{1}{x+1}}$$

$$= \lim_{x\to 0+} -\frac{\frac{1}{x\ln x}}{\frac{1}{(x+1)(\ln(x+1))^2}} = \lim_{x\to 0+} -\frac{\overset{\to 1\cdot 0}{(x+1)(\ln(x+1))^2}}{\underset{\to 0}{x\ln x}} \underset{L'H}{=} \lim_{x\to 0+} -(x+1)\lim_{x\to 0+} \frac{\overset{\to 0}{(\ln(x+1))^2}}{\underset{\to 0}{x\ln x}}$$

$$\underset{L'H}{=} -1\cdot\lim_{x\to 0+} \frac{2\ln(x+1)\cdot\frac{1}{x+1}}{\ln x + 1} = -\lim_{x\to 0+} \frac{2}{x+1} \lim_{x\to 0+} \frac{\ln(x+1)}{\underset{\to -\infty+1}{\ln x + 1}} = -2\cdot\frac{0}{-\infty} = -2\cdot 0 = 0$$