Integration Theory — Exam 1
MAT 725, Fall 2015 — D. Ivanšić

Name: ______________________________

Show all your work!

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) Define the uniform norm of a bounded function.

Theory 2. (3pts) State the theorem on interchange of limit and integral.

Theory 3. (3pts) Define when a function $f : [a, b] \to \mathbf{R}$ is of bounded variation.

Type A problems (5pts each)

A1. Show that the sequence of functions $f_n(x) = \dfrac{2^{-nx}}{1 + nx^2}$ does not converge uniformly to 0 on $[0, 1]$.

A2. Show that $\displaystyle\lim_{n\to\infty} \int_0^1 \sin\left(\frac{x^2}{n}\right)\, dx = 0$.

A3. As previously defined, let $x^a = E(aL(x))$, where E, L are the exponential and logarithmic functions. Show 1) $(xy)^a = x^a y^a$ 2) $L(x^a) = aL(x)$.

A4. Find the variation of the function $f(x) = \sin x$ on the interval $[0, 4\pi]$.

A5. The curve C is given by $\mathbf{r} : [0, 1] \to \mathbf{R}^3$, $\mathbf{r}(t) = (5t, 2t^{\frac{3}{2}}, t^2)$. Show that C is rectifiable and give an upper bound for its length.

Type B problems (8pts each)

B1. Let $f_n : \mathbf{R} \to \mathbf{R}$ be the sequence of functions given by $f_n(x) = \dfrac{1}{1 + n^2x^2}$. Show that
a) (f_n) converges pointwise to a function f.
b) (f_n) converges uniformly on $[a, \infty)$ for every $a > 0$.
c) (f_n) does not converge uniformly on $[0, \infty)$.

B2. Find a rational number (it doesn't have to be simplified to form $\frac{m}{n}$) that approximates $\sqrt[3]{e}$ with accuracy 10^{-4}.

B3. Suppose $f : \mathbf{R} \to \mathbf{R}$ has the properties: $f''(x) = \frac{1}{2} f(x)$, and $f(0) = f'(0) = 0$. Show that $f(x) = 0$ for all $x \in \mathbf{R}$.

B4. Show: if $f : [a, b] \to \mathbf{R}$ is of bounded variation, then so is f^2, where $f^2(x) = (f(x))^2$.

B5. Suppose $g : \mathbf{R} \to \mathbf{R}$ has a continuous derivative, and $f : [a, b] \to \mathbf{R}$ is of bounded variation. Show that $g \circ f : [a, b] \to \mathbf{R}$ is of bounded variation.

Type C problems (12pts each)

C1. Does the converse to **B4** hold: if f^2 is of bounded variation, is f?

Integration Theory — Exam 2 **MAT 725, Fall 2015 — D. Ivanšić**	**Name:** ____________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) Define a Riemann-Stieltjes sum of f with respect to ϕ for functions $f, \phi : [a, b] \to \mathbf{R}$. Then state when f has Riemann Stieltjes integral L with respect to ϕ.

Theory 2. (3pts) Define a point of closure of a set and the closure of a set.

Theory 3. (3pts) Define the Lebesgue outer measure of a set $A \subseteq \mathbf{R}$.

Type A problems (5pts each)

A1. Calculate $\int_0^3 (2x + 1)\, dx^2$.

A2. Use the definition to show that the set $[1, \infty)$ is not compact.

A3. Let $A = \cup_{k=1}^{\infty} (\frac{1}{2k}, \frac{1}{2k-1})$ (union of intervals). Determine $\operatorname{Int} A$ and $\overline{A}$ with explanation.

A4. Show that a closed interval $[a, b]$ is a G_δ-set, and an open interval (a, b) is an F_σ-set.

A5. Let $A = \{\frac{m}{2^n} \mid m, n \in \mathbf{Z}, m \text{ odd}, n \geq 0\}$. What is $m^* A$? Justify. (m^* = outer measure)

A6. Let $m : \mathcal{A} \to [0, \infty]$ be a measure, $A, B \in \mathcal{A}$ and $A \subseteq B$. If $mA = 0$, show that $mB = m(B - A)$.

Type B problems (8pts each)

B1. Show: if $\int_a^b f\, d\phi$ exists, and $a < c < b$, then $\int_a^c f\, d\phi$ exists.

B2. Let $f : [a, b] \to \mathbf{R}$ be a continuous function of bounded variation. Justify why $\int_a^b f\, df$ exists and use Riemann-Stieltjes integration by parts to find it.

B3. Show that for any two sets $A, B \subseteq \mathbf{R}$, $\operatorname{Int}(A \cap B) = \operatorname{Int} A \cap \operatorname{Int} B$.

B4. What is the closure of the set A from **A5**? Justify precisely.

B5. Let $A = \mathbf{Q} \cap [0, 1]$. Show that A is not compact and find a sequence in A that does not have a subsequence which converges to an element of A.

B6. Let $\mathcal{F} = \{(a, \infty) \mid a \in \mathbf{R}\}$. Show that the smallest σ-algebra that contains $\mathcal{F}$ is the Borel sets.

B7. Let $m : \mathcal{A} \to [0, \infty]$ be a measure, $A, B \in \mathcal{A}$. Show that $m(A \cup B) = mA + mB - m(A \cap B)$.

Type C problems (12pts each)

C1. Let $\mathcal{A}$ be the collection of all sets $A \subseteq \mathbf{R}$ such that either A or A^c is countable.
a) Show that $\mathcal{A}$ is a σ-algebra (don't forget that we consider $\emptyset$ to be finite).
b) Give an example of a subset of $\mathbf{R}$ that is not in $\mathcal{A}$.

Integration Theory — Exam 3 **MAT 725, Fall 2015 — D. Ivanšić**	**Name:** ______________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) Define a measurable set.

Theory 2. (3pts) State the theorem on outer approximation by open sets.

Theory 3. (3pts) State Vitali's theorem on existence of nonmeasurable sets.

Type A problems (5pts each)

A1. Let A be any subset of $\mathbf{R}$. If $-A = \{-x \mid x \in A\}$, show that $m^*(-A) = m^*A$

A2. Let E be measurable and $E \subseteq [0,1]$. Set $E_k = E \cap \left(\frac{1}{k}, 1 - \frac{1}{k}\right)$. What is $\lim\limits_{k\to\infty} mE_k$?

A3. Let $\{E_k \mid k \in \mathbf{N}\}$ be a collection of measurable sets. Show that $\bigcup\limits_{k=1}^{\infty} E_k$ can be written as a disjoint union of measurable sets.

A4. Let E have finite outer measure. Show that E is measurable if and only if there exists an F_σ-set $F \subset E$ such that $m^*F = m^*E$.

A5. Show that a nonmeasurable set must be uncountable.

A6. Show that every nonmeasurable set has a proper subset that is nonmeasurable.

Type B problems (8pts each)

B1. Show that the smallest σ-algebra containing intervals of form $(a,b]$, $a, b < \infty$, is the Borel sets.

B2. Give an example of an open unbounded set that has finite measure.

B3. Suppose E is measurable and has finite measure. Show that for every $\epsilon > 0$ there exists an $n \in \mathbf{N}$ such that $m(E - [-n,n]) < \epsilon$. (In other words, most of E's measure is in a bounded interval.)

B4. We have shown in that for any set E with finite outer measure there is a G_δ-set $G \supseteq E$ such that $m^*G = m^*E$. Use this fact to show that when E is not measurable, it fails the definition using a set A that may be taken to be a G_δ-set.

B5. Show that a union of two measurable sets is measurable either by definition or by using inner approximation by closed sets.

B6. Show: if E is measurable and has finite measure, then for every $\epsilon > 0$, E is union of finitely many measurable sets of measure $< \epsilon$.

Type C problems (12pts each)

C1. Let E be a measurable set with finite measure M. Define $f : \mathbf{R} \to [0, M]$ by setting $f(x) = m((-\infty, x] \cap E)$.
a) Show that f is increasing and the codomain is correct.
b) Show that f is continuous — don't do anything difficult, because it's not.
c) Show that the range of f contains the open interval $(0, M)$. In particular, there exists a set whose measure is $M/2$.
d) Give an example of a set E where the range equals $(0, M)$, so $f(x) \neq 0, M$ for all $x \in \mathbf{R}$.